The Theory of Interest

Second Edition

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To Toni, Matt and Lexi

Preface

This second edition is a substantially revised and expanded treatment of the theory of interest from that contained in the first edition. With a few minor exceptions, all the material in the first edition has been retained and updated. In addition, a significant amount of new material has been added.

In addition to a thorough treatment of the mathematical theory of interest, the second edition also introduces the reader to the economic and financial theory of interest. The effect of such factors as inflation, risk and uncertainty, and yield curves is examined. In addition, material has been added to introduce a number of modern financial instruments that have become important since the first edition.

The interrelationship between assets and liabilities has received increased attention in recent years. The second edition provides the reader with an introduction to the tools available to quantify and manage this relationship. The techniques of duration analysis, immunization, dedication, and scenario testing are considered at a level commensurate with the rest of the book.

The first edition adopted a largely deterministic approach to the subject of interest. The second edition retains this approach in the earlier chapters, but introduces a number of stochastic approaches toward the end. Models in which successive rates of interest are independent and dependent random variables are examined. This is followed with a discussion of the Capital Asset Pricing Model, including an important extension of the model to liabilities which is often not considered in other literature. Finally, two contemporary approaches to valuing options are presented; namely, the Black-Scholes formula and binomial lattices.

The material on yield rates in the first edition has been significantly expanded. New topics added include time-weighted rates of interest and the distinction between portfolio and investment year methods of calculating interest. Calculation reflecting reinvestment rates have been expanded and

stressed throughout the entire book. Discounted cash flow analysis and capital budgeting as decision tools have been given more emphasis. Finally, the second edition contains a more thorough treatment of non-unique yield rates including various ways to resolve the difficulties they create.

The material on amortization of loans has been significantly expanded. A section on real estate mortgages has been added, including an illustrative 30-year amortization schedule. New types of mortgage loans are defined and analyzed, such as the adjustable rate mortgage. The discussion of truth in lending requirements is more complete than in the first edition. Finally, new material has been added on amortization with step-rate amounts of principal and on continuous amortization.

The second edition contains several other extensions to the material in the first edition. A sounder theoretical foundation for compound interest and simple interest is presented at the beginning of the book. A section has been added exploring the pitfalls of attempting to value annuities at simple interest. The approach used to value annuities payable at a different frequency than interest is convertible has been modified. More emphasis is placed on annuity calculations involving varying interest and on annuities with payments varying in geometric progression. The material on bonds has been expanded to include the peculiar algorithm widely used in practice to value bonds between coupons payment dates, which is a mixture of simple and compound interest. Finally, illustrations have been added on such "real-world" practices as penalties for early withdrawal and prepayment penalties.

The computational approaches in the book have been modernized to reflect the widespread availability of pocket calculators with exponential and logarithmic functions. In a few instances, results are obtained which require a calculator with built-in financial functions, but these have been kept to a minimum. Compound interest tables are still included and have been extended to include higher rates of interest. Iteration techniques are used in solving for unknown rates of interest throughout the book.

The pedagogical approach of the second edition is similar to the first edition. Verbal interpretations of key results are emphasized throughout. Illustrative examples are provided at the end of most sections. The number of exercises has been expanded substantially to 480. Each exercise is intended to illustrate a somewhat different point, so that the number of repetitious exercises has been kept to a minimum. Many of the exercises involve more realistic rates of interest than in the first edition. Answers to the exercises are provided at the back of the book.

A working knowledge of basic calculus is required, since the continuous nature of interest is recognized throughout the book. A background in numerical analysis will be helpful in connection with the iteration techniques, although the reader without such a background can still follow the development. The new sections on stochastic approaches to interest assume a knowledge of basic statistics.

The book is designed to be appropriate for both classroom use with an instructor and for self-study by those learning the subject without the aid of an instructor. The amount of material probably exceeds that which can be covered thoroughly in a one-semester university course.

The second edition contains a total of 10 appendices. These appendices cover topics which may be of value to certain readers, but are not of sufficiently broad interest to warrant inclusion in the chapters. An extensive bibliography has been added at the end of the book. Also, a glossary of notation has been added indicating the section in which each new symbol has been introduced.

The author is indebted to a number of correspondents who have written over the years in connection with first edition. As a result of this correspondence, a number of improvements have been incorporated into the second edition.

The author also wishes to express his appreciation to the Society of Actuaries for using both editions as the syllabus reference on the actuarial examination covering the mathematics of compound interest. A committee of the Society of Actuaries revised a working draft of the second edition and as a result a number of very helpful suggestions were received which resulted in a significant improvement in the final manuscript.

The author wishes to give special recognition and appreciation to Tabitha Miller who spent uncountable hours in typing the manuscript for the book. Since final copy was produced by word processing rather than by typesetting, this was a very painstaking and often frustrating process. The author will forever be in her debt for the many months of arduous effort in producing this book.

Finally, the author wishes to recognize the patience and understanding of his wife Toni for her unfailing support. Without her dedication and support this book could not have been completed. She bore a very special burden, since much of the work on the book was completed during a period in which we adopted two lovely children.

April 1991 STEPHEN G. KELLISON

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1

The measurement of interest

1.1 INTRODUCTION

Interest may be defined as the compensation that a borrower of capital pays to a lender of capital for its use. Thus, interest can be viewed as a form of rent that the borrower pays to the lender to compensate for the loss of use of the capital by the lender while it is loaned to the borrower. In theory, capital and interest need not be expressed in terms of the same commodity. For example, Farmer A may lend a tractor to Farmer B for use in harvesting B's wheat crop in return for a percentage of the wheat harvested. In this example, the tractor is capital and the portion of wheat that B gives to A is interest. However, for almost all applications, both capital and interest are expressed in terms of money.

In Chapter 1 the various quantitative measures of interest are analyzed. This chapter includes most of the basic principles involved in the measurement of interest. Chapters 2 through 8 elaborate and extend these basic principles to more complex financial transactions. These chapters explore the various methods by which interest is calculated and by which capital and interest are repaid by the borrower to the lender.

Chapters 1-8 in essence are concerned with the *mathematical* theory of interest on a deterministic basis. Chapter 9 introduces the student to the *economic and financial* theory of interest and related topics. Finally, Chapter 10 approaches the subject of interest from a *stochastic* rather than *deterministic* approach.

1.2 THE ACCUMULATION AND AMOUNT FUNCTIONS

A common financial transaction is the investment of an amount of money at interest. For example, a person may invest in a savings account at a bank.

The initial amount of money (capital) invested is called the *principal* and the total amount received after a period of time is called the *accumulated value*. The difference between the accumulated value and the principal is the *amount of interest*, or just *interest*, earned during the period of investment.

For the moment, assume that given the original principal invested, the accumulated value at any point in time can be determined. We will assume that no principal is added or withdrawn during the period of investment, i.e. that any change in the fund is due strictly to the effect of interest. Later we will relax this assumption and allow for contributions and withdrawals during the period of investment.

Let t measure time from the date of investment. In theory, time may be measured in many different units, e.g., days, months, decades, etc. The unit in which time is measured is called the *measurement period*, or just *period*. The most common measurement period is one year, and this will be assumed unless stated otherwise.

Consider the investment of one unit of principal. We can define an *accumulation function* $a(t)$ which gives the accumulated value at time $t \geq 0$ of an original investment of 1.

What properties does this function possess?

1. It is clear that $a(0) = 1$.
2. $a(t)$ is generally an increasing function. A decrease in the functional values for increasing t would imply negative interest. Although negative interest is possible mathematically, it is not relevant to most situations encountered in practice. However, there are situations in which negative interest does appear, e.g. an investment fund which loses money over a certain period of time. Constant functional values would imply zero interest, a situation occurring occasionally.
3. If interest accrues continuously, as is usually the case, the function will be continuous. However, there are situations in which interest does not accrue continuously between interest payment dates, in which case $a(t)$ possesses discontinuities.

In general, the original principal invested will not be one unit but will be some amount $k > 0$. We now define an *amount function* $A(t)$ which gives the accumulated value at time $t \geq 0$ of an original investment of k. Then we have

$$A(t) = k \cdot a(t) \tag{1.1}$$

and

$$A(0) = k.$$

The second and third properties of $a(t)$ listed above clearly also hold for $A(t)$.

We will denote the amount of interest earned during the nth period from the date of investment by I_n. Then

$$I_n = A(n) - A(n-1) \text{ for integral } n \geq 1. \tag{1.2}$$

It should be noted that I_n involves the effect of interest over an interval of time, whereas $A(n)$ is an amount at a specific point in time.

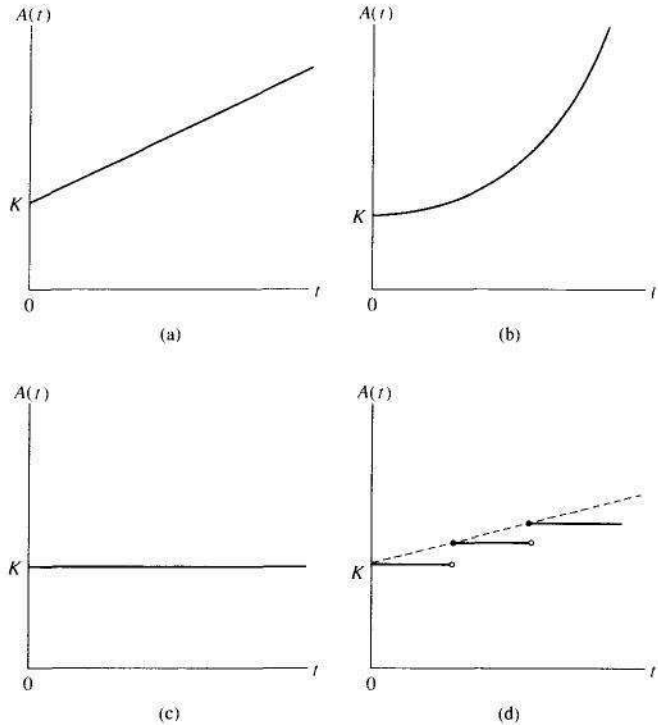

Figure 1.1 Four illustrative amount functions

Actually, the accumulation function is a special case of the amount function for which $k = 1$. However, the accumulation function will be significant enough in the rest of this chapter to warrant a separate definition. In many cases, the accumulation function and the amount function can be used interchangeably.

Figure 1.1 shows four examples of amount functions. Figure (*a*) is a linear amount function. Figure (*b*) is nonlinear, in this case an exponential curve. Figure (*c*) is an amount function which is horizontal, i.e. the slope is zero. This figure represents an amount function in which the principal is accruing no interest. Figure (*d*) is an amount function in which interest is not accruing

continuously but is accruing in discrete segments with no interest accruing between interest payment dates.

In the following sections, various measures of interest will be developed from the accumulation function. In practice, two particular accumulation functions will handle most situations which arise. However, the reader should understand the properties of a general accumulation function as defined in this section and be able to work with it.

1.3 THE EFFECTIVE RATE OF INTEREST

The first measure of interest is called the *effective rate of interest* and is denoted by i. A precise definition is:

The effective rate of interest i is the amount of money that one unit invested at the beginning of a period will earn during the period, where interest is paid at the end of the period.

Note that in terms of the accumulation function, this definition is equivalent to saying that

$$i = a(1) - a(0)$$

or

$$a(1) = 1 + i. \tag{1.3}$$

Several observations about this definition are important:

1. The use of the word "effective" is not intuitively clear. This term is used for rates of interest in which interest is paid once per measurement period. This will be contrasted with "nominal" rates of interest, in which interest is paid more frequently than once per measurement period, to be considered in Section 1.8.
2. The effective rate of interest is often expressed as a percentage, e.g. $i = 8\%$. The concept of the effective rate of interest as a percentage is not inconsistent with the definition above, which states that it is an amount of money, since 8% can be looked upon as .08 per unit of principal.
3. The amount of principal remains constant throughout the period, i.e. no new principal is contributed and no principal is withdrawn during the period.
4. The effective rate of interest is a measure in which interest is paid at the end of the period. The significance of this statement is not immediately clear, but it will become evident in Section 1.7, where a situation is described in which interest is paid at the beginning of the period.

The effective rate of interest can be defined in terms of the amount function as follows:

$$i = \frac{(1+i)-1}{1} = \frac{a(1)-a(0)}{a(0)} = \frac{A(1)-A(0)}{A(0)} = \frac{I_1}{A(0)}. \qquad (1.4a)$$

Thus, an alternate definition is:

The effective rate of interest i is the ratio of the amount of interest earned during the period to the amount of principal invested at the beginning of the period.

The same four observations made above also apply to this alternate definition.

Effective rates of interest can be calculated over any measurement period. Let i_n be the effective rate of interest during the nth period from the date of investment. Then we have

$$i_n = \frac{A(n)-A(n-1)}{A(n-1)} = \frac{I_n}{A(n-1)} \text{ for integral } n \geq 1. \qquad (1.4b)$$

Within this framework, the "i" in formula (1.4a) might more properly be labeled i_1.

Although formula (1.4b) allows the various effective rates of interest i_n to vary for different n, it will be demonstrated in Section 1.5 that for one very important accumulation function, the effective rate of interest is constant over successive measurement periods, i.e. for all integral $n \geq 1$.

1.4 SIMPLE INTEREST

It was shown in the preceding sections that $a(0) = 1$ and $a(1) = 1 + i$. There are an infinite number of accumulation functions that pass through these two points. Two of these are most significant in practice. The first, simple interest, will be discussed in this section; and the second, compound interest, will be discussed in Section 1.5.

Consider the investment of one unit such that the amount of interest earned during each period is constant. The accumulated value of 1 at the end of the first period is $1 + i$, at the end of the second period it is $1 + 2i$, etc. Thus, in general, we have a linear accumulation function

$$a(t) = 1 + it \text{ for integral } t \geq 0. \qquad (1.5)$$

The accruing of interest according to this pattern is called *simple interest*.

It can be shown that a constant rate of simple interest does not imply a constant effective rate of interest. Let i be the rate of simple interest and let i_n be the effective rate of interest for the nth period, as defined in Section 1.3. Then we have

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{[1+in] - [1+i(n-1)]}{1+i(n-1)} = \frac{i}{1+i(n-1)} \qquad (1.6)$$

for integral $n \geq 1$, which is a decreasing function of n. Thus, a constant rate of simple interest implies a decreasing effective rate of interest.

The accumulation function for simple interest has been defined only for integral values of $t \geq 0$. However, it is natural to extend the definition to nonintegral values of $t > 0$ as well. This is equivalent to the crediting of interest proportionally over any fraction of a period. If this is the case, then the amount function can be represented by Figure 1.1(*a*). If interest is accrued only for completed periods with no credit for fractional periods, then the amount function becomes a step function with discontinuities as illustrated by Figure 1.1 (*d*). Unless stated otherwise, it will be assumed that interest is accrued proportionally over fractional periods under simple interest.

A more rigorous mathematical approach to the definition of $a(t)$ for nonintegral values of t can be developed by starting with the following property we would want simple interest to possess:

$$a(t+s) = a(t) + a(s) - 1 \text{ for } t \geq 0 \text{ and } s \geq 0. \tag{1.7}$$

In essence, formula (1.7) says that under simple interest the amount of interest earned by an initial investment of one unit over $t + s$ periods is equal to the amount of interest earned over t periods plus the amount of interest earned over s periods. The -1 is necessary in formula (1.7), since otherwise there would be an investment of two units on the right-hand side of the equation.

Assuming $a(t)$ is differentiable, from the definition of the derivative we have

$$\begin{aligned} a'(t) &= \lim_{s \to 0} \frac{a(t+s) - a(t)}{s} \\ &= \lim_{s \to 0} \frac{[a(t) + a(s) - 1] - a(t)}{s} \\ &= \lim_{s \to 0} \frac{a(s) - 1}{s} \\ &= \lim_{s \to 0} \frac{a(s) - a(0)}{s} \\ &= a'(0), \text{ a constant.} \end{aligned}$$

Replacing t by r and integrating both sides between the limits 0 and t, we have

$$\begin{aligned} \int_0^t a'(r)dr &= \int_0^t a'(0)dr \\ a(t) - a(0) &= t \cdot a'(0) \\ a(t) &= 1 + t \cdot a'(0). \end{aligned}$$

If we let $t = 1$ and remember that $a(1) = 1 + i$, we have

$$a(1) = 1 + i = 1 + a'(0)$$

so that $a'(0) = i$. Thus, substituting back we have the familiar result

$$a(t) = 1 + it \text{ for } t \geq 0. \tag{1.5}$$

The above derivation does not depend on t being a positive integer, and is valid for all $t \geq 0$.

***Example 1.1** Find the accumulated value of \$2000 invested for four years, if the rate of simple interest is 8% annum.*

The answer is

$$2000[1 + (.08)(4)] = \$2640.$$

Note that the amount of interest earned is 2640 - 2000 = \$640. This could also have been obtained as 2000(.08)(4), or, in general, as $A(0) \cdot it$. In different notation, this becomes the familiar result from elementary and secondary school

$$I = Prt$$

which states that the amount of interest is equal to the product of the amount of principal, the rate of interest, and the period of time.

1.5 COMPOUND INTEREST

Simple interest has the property that the interest is not reinvested to earn additional interest. For example, consider an investment of \$100 for two years at 10% simple interest. Under simple interest the investor will receive \$10 at the end of each of the two years. However, in reality, for the second year the investor has \$110 which could have been invested. Clearly, it would be advantageous to invest the \$110 at 10%, since the investor would then receive \$11 in interest for the second year instead of \$10.

The theory of *compound interest* handles this problem by assuming that the interest earned is automatically reinvested. The word "compound" refers to the process of interest being reinvested to earn additional interest. With compound interest the total investment of principal and interest earned to date is kept invested at all times.

It is now desired to find the accumulation function for compound interest. Consider the investment of 1 which accumulates to $1 + i$ at the end of the first period. This balance of $1 + i$ can be considered as principal at the beginning of the second period and will earn interest of $i(1 + i)$ during the second period. The balance at the end of the second period is $(1 + i) + i(1 + i) = (1 + i)^2$. Similarly, the balance of $(1 + i)^2$ can be considered as principal at the beginning of the third period and will earn interest of $i(1 + i)^2$ during the third period. The balance at the end of the third period is $(1 + i)^2 + i(1 + i)^2 = (1 + i)^3$. Continuing this process indefinitely, we obtain

$$a(t) = (1 + i)^t \text{ for integral } t \geq 0. \tag{1.8}$$

It can be shown that a constant rate of compound interest implies a constant effective rate of interest and, moreover, that the two are equal. Let i be the rate of compound interest and let i_n be the effective rate of interest for the nth period, as defined in Section 1.3. Then, we have

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{(1+i)^n - (1+i)^{n-1}}{(1+i)^{n-1}} = \frac{(1+i) - 1}{1} = i$$

which is independent of n. Thus, although defined differently, a rate of compound interest and an effective rate of interest are identical.

The result just derived can be compared with the result obtained in Section 1.4; namely, that a constant rate of simple interest implies a decreasing effective rate of interest. This result should be intuitively clear, since simple interest becomes progressively less favorable to the investor as the period of investment increases.

The accumulation function for compound interest has been defined only for integral values of $t \geq 0$. It is necessary to develop the accumulation function for nonintegral values of $t > 0$. It is possible to address this question on a basis analogous to that used in Section 1.4 for simple interest.

We start with the following property we would want compound interest to possess:

$$a(t+s) = a(t) \cdot a(s) \text{ for } t \geq 0 \text{ and } s \geq 0. \tag{1.9}$$

In essence, formula (1.9) says that under compound interest the amount of interest earned by an initial investment of one unit over $t + s$ periods is equal to the amount of interest earned if the investment is terminated at the end of t periods and the accumulated value at that point is immediately reinvested for an additional s periods.

Assuming $a(t)$ is differentiable, from the definition of the derivative we have

$$\begin{aligned} a'(t) &= \lim_{s \to 0} \frac{a(t+s) - a(t)}{s} \\ &= \lim_{s \to 0} \frac{a(t) \cdot a(s) - a(t)}{s} \\ &= a(t) \lim_{s \to 0} \frac{a(s) - 1}{s} \\ &= a(t) \cdot a'(0). \end{aligned}$$

Thus

$$\frac{a'(t)}{a(t)} = \frac{d}{dt} \log_e a(t) = a'(0).$$

Replacing t by r and integrating both sides between the limits 0 and t, we have

$$\int_0^t \frac{d}{dr} \log_e a(r) dr = \int_0^t a'(0)\, dr$$

$$\log_e a(t) - \log_e a(0) = t \cdot a'(0)$$

$$\log_e a(t) = t \cdot a'(0)$$

since $\log_e a(0) = 0$. If we let $t = 1$ and remember that $a(1) = 1 + i$, we have

$$\log_e a(1) = \log_e (1 + i) = a'(0).$$

Thus, substituting back we have the result

$$\log_e a(t) = t \log_e (1 + i) = \log_e (1 + i)^t$$

or

$$a(t) = (1 + i)^t \text{ for } t \geq 0. \tag{1.8}$$

The above derivation does not depend on t being a positive integer, and is valid for all $t \geq 0$.

Unless stated otherwise, it will be assumed that interest is accrued over fractional periods according to formula (1.8) under compound interest. Thus the amount function is exponential and can be represented by Figure 1.1(*b*). The exponential form of the amount function should not be unexpected, since many growth curves encountered in the natural sciences are exponential.

Some readers may be troubled, on the one hand, by the statement that interest is paid at the end of the period and, on the other hand, by the statement that interest is accruing continuously. At first glance, the two statements appear contradictory. However, there is no inconsistency as long as interest is accrued over fractional periods as well as completed periods. When this is the case, the accumulated values at any point in time are equal from either perspective.

It is clear that simple and compound interest produce the same result over one measurement period. Over a longer period, compound interest produces a larger accumulated value than simple interest while the opposite is true over a shorter period. The proofs of these results are left as exercises.

Another insight into the difference between simple interest and compound interest can be seen from the two growth patterns involved. Under simple interest, it is the *absolute amount* of growth that is constant over equal periods of time; while under compound interest, it is the *relative rate* of growth that is constant. In terms of symbols, under simple interest

$$a(t + s) - a(t)$$

is independent of t, while under compound interest

$$\frac{a(t + s) - a(t)}{a(t)}$$

is independent of t.

Compound interest is used almost exclusively for financial transactions covering a period of one year or more and is often used for shorter term transactions as well. Simple interest is occasionally used for short-term transactions and as an approximation for compound interest over fractional periods. This latter use of simple interest will be examined in more detail in Section 2.2. Hereafter, unless stated otherwise, we will use compound interest instead of simple interest.

There is an implicit assumption in this section that the interest earned under compound interest is reinvested at the same rate as the original investment. Although this is often the case, many situations do arise in practice in which the interest earned is reinvested at a different rate. Section 5.4 contains an analysis of the results when the reinvestment rate for the interest earned differs from the rate on the original principal.

Example 1.2 Rework Example 1.1 using compound interest instead of simple interest.

The answer is

$$2000(1.08)^4 = \$2720.98.$$

This answer is in contrast with the answer of \$2640 using simple interest. The extra \$80.98 is the result of compounding of interest.

1.6 PRESENT VALUE

We have seen that an investment of 1 will accumulate to $1 + i$ at the end of one period. The term $1 + i$ is often called an *accumulation factor*, since it accumulates the value of an investment at the beginning of a period to its value at the end of the period.

It is often necessary to determine how much a person must invest initially so that the balance will be 1 at the end of one period. The answer is $(1 + i)^{-1}$, since this amount will accumulate to 1 at the end of one period. We now define a new symbol v, such that

$$v = \frac{1}{1+i}. \tag{1.10}$$

The term v is often called a *discount factor*, since it "discounts" the value of an investment at the end of a period to its value at the beginning of the period.

We can generalize the above result to periods of time other than one period, i.e. to find the amount which a person must invest in order to accumulate an amount of 1 at the end of t periods. The answer is the reciprocal of the accumulation function $a^{-1}(t)$, since the accumulated value of this amount at the end of t periods is $a^{-1}(t) \cdot a(t) = 1$. We will call $a^{-1}(t)$ the *discount function*.

Thus, we obtain the following results for $t \geq 0$:

$$\text{Simple Interest:} \qquad a^{-1}(t) = \frac{1}{1+it} \tag{1.11}$$

$$\text{Compound Interest:} \qquad a^{-1}(t) = \frac{1}{(1+i)^t} = v^t \tag{1.12}$$

As specified before, we will use compound interest unless stated otherwise.

In a sense, *accumulating* and *discounting* are opposite processes. The term $(1+i)^t$ is said to be the *accumulated value* of 1 at the end of t periods. The term v^t is said to be the *present value* (or *discounted value*) of 1 to be paid at the end of t periods.

The term "accumulated value" as defined above seems to refer strictly to payments made in the past, while the term "present value" seems to refer strictly to payments to be made in the future. This is the sense in which we will use the terms. Some writers have used "present value" to refer to either past or future payments. We will use the term *current value* for this purpose.

It is interesting to relate v^t to the accumulation function for compound interest from an alternate viewpoint. It is immediately clear that the values of v^t extend the definition of the accumulation function to negative values of t. Thus, the accumulation function for compound interest has meaning for all values of t. The graph of this function is shown in Figure 1.2.

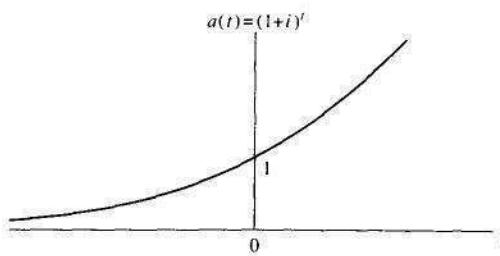

Figure 1.2 Accumulation function for compound interest

Example 1.3 Find the amount which must be invested at a rate of simple interest of 9% per annum in order to accumulate $1000 at the end of three years.

The answer is

$$\frac{1000}{1+(.09)(3)} = \frac{1000}{1.27} = \$787.40.$$

Example 1.4 Rework Example 1.3 using compound interest instead of simple interest.

The answer is

$$1000v^3 = \frac{1000}{(1.09)^3} = \$772.18.$$

From general reasoning, the reader should justify the relative magnitude of the answers to Examples 1.3 and 1.4.

1.7 THE EFFECTIVE RATE OF DISCOUNT

In Section 1.3 the effective rate of interest was defined as a measure of interest paid at the end of the period. In this section we define the *effective rate of discount*, denoted by d, as a measure of interest paid at the beginning of the period.

A numerical illustration will help make this distinction clear. If A goes to a bank and borrows \$100 for one year at an effective rate of interest of 6%, then the bank will give A \$100. At the end of the year, A will repay the bank the original loan of \$100, plus interest of \$6, or a total of \$106.

However, if A borrows \$100 for one year at an effective rate of discount of 6%, then the bank will collect its interest of 6% in advance and will give A only \$94. At the end of the year, A will repay \$100.

Thus, it is clear that an effective rate of interest of 6% is not the same as an effective rate of discount of 6%. In the above example, A paid \$6 interest in both cases. However, in the case of interest paid at the end of the year, A had the use of \$100 for the year, while in the case of interest paid at the beginning of the year, A had the use of only \$94 for the year.

Looking at the above illustration in a slightly different manner, in the case of an effective rate of interest, the 6% is taken as a percentage of the balance at the beginning of the year, while in the case of an effective rate of discount, the 6% is taken as a percentage of the balance at the end of the year. Thus, we can formulate a precise definition of the effective rate of discount as follows:

> *The effective rate of discount d is the ratio of the amount of interest (sometimes called the "amount of discount" or just "discount") earned during the period to the amount invested at the end of the period.*

This definition is analogous to the alternate definition of the effective rate of interest given in Section 1.3.

Several observations about the above definition are important:

1. Observations 1, 2, and 3 in Section 1.3 on the definition of the effective rate of interest also apply to the definition of the effective rate of discount.
2. The phrases *amount of discount* and *amount of interest* can be used interchangeably in situations involving rates of discount.

3. The definition does not use the word "principal," since the definition of principal refers to the amount invested at the beginning of the period and not at the end of the period.
4. The key distinction between the effective rate of interest and the effective rate of discount can be summarized as follows:
   a) Interest--paid at the end of the period on the balance at the beginning of the period.
   b) Discount--paid at the beginning of the period on the balance at the end of the period.

Some readers find the use of the word "paid" in connection with rates of discount somewhat confusing, since the borrower does not directly "pay" the interest as with rates of interest. However, the net result of deducting the interest in advance is no different than if the full amount is borrowed and then the borrower immediately pays the interest.

In fact, some readers may find the use of the word "paid" confusing in another sense as well. This confusion involves the possible implication that interest is "paid" in installments as it is earned rather than being accumulated to earn additional interest. We do not attach such a connotation to the word "paid," which can be used in either context above. A term such as "credited" might be preferred by those who find the word "paid" ambiguous in this sense.

Effective rates of discount can be calculated over any particular measurement period. Let d_n be the effective rate of discount during the nth period from the date of investment. A formula analogous to (1.4*b*) is

$$d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{I_n}{A(n)} \text{ for integral } n \geq 1. \tag{1.13}$$

As mentioned above, I_n may be commonly called either the "*amount of discount*" or the "*amount of interest.*" In general, d_n may vary from period to period. However, if we have compound interest, in which case the effective rate of interest is constant, then the effective rate of discount is also constant. The proof of this statement is left as an exercise. These situations are referred to as *compound discount*, a term analogous to "compound interest."

The illustration discussed earlier in this section showed that an effective rate of interest of 6% is not the same as an effective rate of discount of 6%. However, there is a definite relationship between effective rates of interest and effective rates of discount.

To develop this relationship we need to define a concept of *equivalency* as follows:

Two rates of interest or discount are said to be equivalent if a given amount of principal invested for the same length of time at each of the rates produces the same accumulated value.

As we shall see in Section 1.8, this definition is applicable for nominal rates of interest and discount, as well as effective rates.

Assume that a person borrows 1 at an effective rate of discount d. Then, in effect, the original principal is $1 - d$ and the amount of interest (discount) is d. However, from the basic definition of i as the ratio of the amount of interest (discount) to the principal, we obtain

$$i = \frac{d}{1-d}. \tag{1.14}$$

This formula expresses i as a function of d.

By simple algebra, it is possible to express d as a function of i

$$\begin{aligned} i &= \frac{d}{1-d} \\ i - id &= d \\ d(1+i) &= i \\ d &= \frac{i}{1+i}. \end{aligned} \tag{1.15a}$$

Formula (1.15*a*) is obvious, since it is merely a restatement of the definition of the effective rate of discount as the ratio of the amount of interest (discount) that 1 will earn during the period to the amount invested at the end of the period.

There are important relationships between d, a rate of discount, and v, a discount factor. One relationship is identical to (1.15*a*)

$$d = iv. \tag{1.15b}$$

This relationship has an interesting verbal interpretation. Interest earned on an investment of 1 paid at the beginning of the period is d. Interest earned on an investment of 1 paid at the end of the period is i. Therefore, if we discount i from the end of the period to the beginning of the period with the discount factor v, we obtain d.

There is another relationship between d and v which is often useful

$$\begin{aligned} d &= \frac{i}{1+i} \\ &= \frac{1+i}{1+i} - \frac{1}{1+i} \\ &= 1 - v. \end{aligned} \tag{1.16}$$

This relationship also can be interpreted verbally. Written in the form $v = 1 - d$, it is immediately seen that both sides of the equation represent the present value of 1 to be paid at the end of the period.

There is one other relationship between i and d which is significant:

$$\begin{aligned} d &= iv \\ &= i(1 - d) \\ &= i - id \end{aligned}$$

or

$$i - d = id. \tag{1.17}$$

This relationship also has an interesting verbal interpretation. A person can either borrow 1 and repay $1 + i$ at the end of the period or borrow $1 - d$ and repay 1 at the end of the period. The expression $i - d$ is the difference in the amount of interest paid. This difference arises because the principal borrowed differs by d. Interest on amount d for one period at rate i is id.

The effective rate of discount, or compound discount, assumes compound interest. However, it is possible to define *simple discount* in a manner analogous to the definition of simple interest. Consider a situation in which the amount of discount earned during each period is constant. Then, the original principal which will produce an accumulated value of 1 at the end of t periods is

$$a^{-1}(t) = 1 - dt \quad \text{for} \quad 0 \leq t < 1/d. \tag{1.18}$$

The second part of the inequality is necessary to keep $a^{-1}(t) > 0$. This contrasts with compound discount, in which case the present value is

$$a^{-1}(t) = v^t = (1 - d)^t \text{ for } t \geq 0. \tag{1.19}$$

It should be noted that formulas (1.14), (1.15), (1.16), and (1.17) assume effective rates of interest and discount and are not valid for simple rates of interest and discount unless the period of investment happens to be exactly one period.

The reader is cautioned that simple discount is not the same as simple interest. However, simple discount does have properties analogous, but opposite, to simple interest. The proofs of the following are left as exercises:

1. A constant rate of simple interest implies a decreasing effective rate of interest, as the period of investment increases, while a constant rate of simple discount implies an increasing effective rate of discount (and interest).
2. Simple and compound discount produce the same result over one measurement period. Over a longer period, simple discount produces a smaller present value than compound discount, while the opposite is true over a shorter period.

It is instructive to illustrate these results graphically. Figure 1.3(*a*) compares the accumulation function under simple interest and compound interest. Similarly, Figure 1.3(*b*) compares the discount function under simple discount and compound discount.

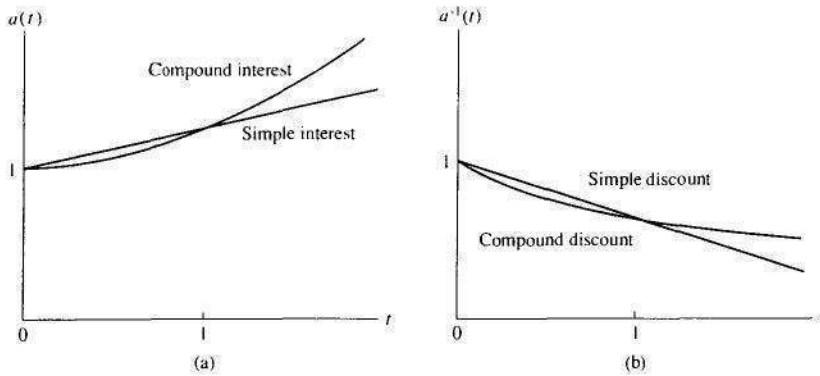

Figure 1.3 Comparisons of: (a) simple and compound interest, and (b) simple and compound discount

Simple discount is used only for short-term transactions and as an approximation for compound discount over fractional periods. It is not as widely used as simple interest.

The word "discount" unfortunately is used in two different contexts with various shades of meaning in each. It is used in connection with present values (discount factor, discount function, discounting, discounted value) and in connection with interest paid at the beginning of the period (effective rate of discount, amount of discount, compound discount, simple discount).

Unfortunately, terms involving the word "discount" are often misused in practice. For example, in the process of discounting, i.e. taking present values, the phrase "rate of discount" is often used when really "rate of interest" is correct. Yet another usage of the word "discount" will appear in Chapter 7. The term will be used to refer to bonds for which the price is less than the redemption value.

Exacerbating the confusion even further is a fourth use of the word "discount" to refer to price reductions. Although we do not attach this meaning to the word in this book, it commonly has this meaning in business and financial transactions. Needless to say, the reader should be careful in using the term "discount" to keep the meaning completely clear and should not hesitate to seek clarification from others using the term if there is any possibility of ambiguity.

Example 1.5 Rework Example 1.3 using simple discount instead of simple interest.

The answer is

$$1000[1 - (.09)(3)] = \$730.$$

Example 1.6 Rework Example 1.3 using compound discount instead of simple interest.

The answer is

$$1000(.91)^3 = \$753.57.$$

From general reasoning, the reader should justify the relative magnitudes of the answers to Examples 1.5 and 1.6.

1.8 NOMINAL RATES OF INTEREST AND DISCOUNT

In Sections 1.3 and 1.7 effective rates of interest and discount were discussed. The term "effective" is used for rates of interest and discount in which interest is paid once per measurement period, either at the end of the period or at the beginning of the period, as the case may be. In this section, we consider situations in which interest is paid more frequently than once per measurement period. Rates of interest and discount in these cases are called "nominal."

Most persons have encountered such situations in practice. For example, Lender A might charge an annual effective rate of 9% on loans; Lender B might charge 8¾ % compounded quarterly; and Lender C might charge 8½% payable in advance and convertible monthly. Most persons probably realize that these rates are not directly comparable, but they would probably not be able to make a valid comparison among them.

Lender A is charging an annual effective rate of interest, which has already been discussed. However, Lender B is charging what we call a nominal rate of interest, while Lender C is charging what we call a nominal rate of discount.

Various terms are used in practice to describe situations in which interest is paid more frequently than once per measurement period. Among these are "payable," "compounded," and "convertible," as in "payable quarterly," "compounded semiannually," and "convertible monthly." The frequency with which interest is paid and reinvested to earn additional interest is called the *interest conversion period.*

The three terms "payable," "compounded," and "convertible" are often used interchangeably. However, they do have different connotations to some users. For example, the term "compounded" seems to imply that the interest being earned is reinvested to earn additional interest, while the term "payable" seems to imply instead that the interest is paid out in installments as it is earned. The term "convertible" does not seem to possess either connotation. The reader is

cautioned not to rely on such connotations when encountering these terms, but to ascertain exactly how the interest is computed and paid.

This section defines nominal rates of interest and discount and develops a systematic method of finding effective and nominal rates of interest and discount which are equivalent. The definition of "equivalency" was given in Section 1.7.

The symbol for a *nominal rate of interest* payable m times per period is $i^{(m)}$, where m is a positive integer > 1. By a nominal rate of interest $i^{(m)}$, we mean a rate payable $m$thly, i.e. the rate of interest is $i^{(m)}/m$ for each mth of a period and not $i^{(m)}$. For example, a nominal rate of 8% convertible quarterly does not mean an interest rate of 8% per quarter but rather an interest rate of 2% per quarter. In reality, we can say that a nominal rate of interest of $i^{(m)}$ per period is identical to an effective rate of interest of $i^{(m)}/m$ per mth of a period.

Thus, from the definition of equivalency we have

$$1 + i = \left[1 + \frac{i^{(m)}}{m}\right]^m \tag{1.20a}$$

since each side of the equation gives the accumulated value of 1 invested for one measurement period. Rearranging, we have

$$i = \left[1 + \frac{i^{(m)}}{m}\right]^m - 1, \tag{1.20b}$$

and

$$i^{(m)} = m[(1 + i)^{\frac{1}{m}} - 1]. \tag{1.20c}$$

Figure 1.4 illustrates accumulation at a nominal rate of interest for one measurement period. The diagonal arrows to the right can be interpreted as plus signs and downward arrows as equal signs.

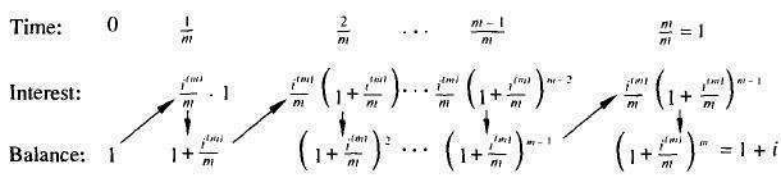

Figure 1.4 Illustration of nominal rates of interest

The symbol for a *nominal rate of discount* payable m times per period is $d^{(m)}$. By a nominal rate of discount $d^{(m)}$, we mean a rate payable $m$thly, i.e. the effective rate of discount is $d^{(m)}/m$ for each mth of a period.

The nominal rate of discount $d^{(m)}$, is a measure of interest paid at the beginning of $m$ths of a period in much the same manner as d is a measure of interest paid at the beginning of the period. By an argument similar to the one used in developing the relationship between $i^{(m)}$ and i, it is possible to develop a formula relating $d^{(m)}$ and d such that they are equivalent.

From the definition of equivalency we have

$$1 - d = \left[1 - \frac{d^{(m)}}{m}\right]^m \tag{1.21a}$$

since each side of the equation gives the present value of 1 to be paid at the end of one measurement period. Rearranging, we have

$$d = 1 - \left[1 - \frac{d^{(m)}}{m}\right]^m \tag{1.21b}$$

and

$$d^{(m)} = m[1 - (1 - d)^{\frac{1}{m}}] = m[1 - v^{\frac{1}{m}}] . \tag{1.21c}$$

Figure 1.5 illustrates discounting at a nominal rate of discount for one measurement period. The diagonal arrows to the left can be interpreted as minus signs and downward arrows as equal signs.

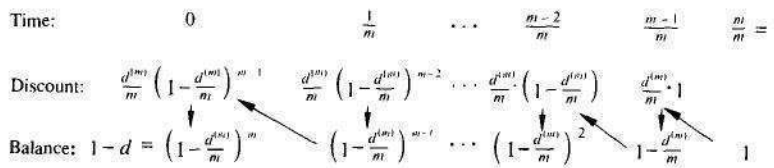

Figure 1.5 Illustration of nominal rates of discount

There is a close relationship between nominal rates of interest and nominal rates of discount. The following relationship holds, since both sides of the equation are equal to $1 + i$,

$$\left[1 + \frac{i^{(m)}}{m}\right]^m = \left[1 - \frac{d^{(p)}}{p}\right]^{-p} . \tag{1.22a}$$

If $m = p$, then formula (1.22a) becomes

$$\left[1 + \frac{i^{(m)}}{m}\right] = \left[1 - \frac{d^{(m)}}{m}\right]^{-1} . \tag{1.22b}$$

If $m = 1$, then $i^{(m)} = i$, the effective rate of interest; and if $p = 1$, then $d^{(p)} = d$, the effective rate of discount. Thus formula (1.22*a*) can be used in general to find equivalent rates of interest or discount, either effective or nominal, convertible with any desired frequency.

Another relationship between $i^{(m)}$ and $d^{(m)}$ which is analogous to formula (1.17) is

$$\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} = \frac{i^{(m)}}{m} \cdot \frac{d^{(m)}}{m}. \tag{1.23}$$

The verbal interpretation of this result is similar to the verbal interpretation of formula (1.17). The derivation is left as an exercise.

In developing the above formulas involving rates of interest and discount which are equivalent, the definition of equivalency given in Section 1.7 was used with a comparison over one measurement period. Although the use of one measurement period was arbitrary, the reader should verify that under compound interest and discount the rates that are equivalent do not depend on the period of time chosen for the comparison. However, for other patterns of interest development, such as simple interest and simple discount, the rates that are equivalent will depend on the period of time chosen for the comparison.

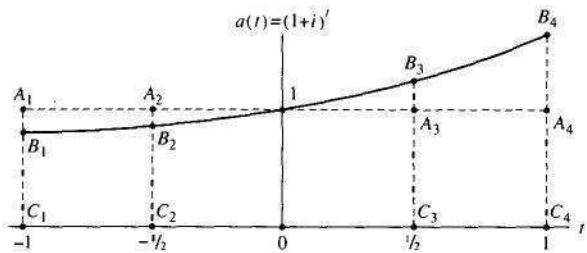

Figure 1.6 Accumulation function illustrating nominal rates of interest and discount

It is instructive to relate nominal rates of interest and discount to the accumulation function $a(t)$. An example is given in Figure 1.6 for $m = 2$. The reader is encouraged to construct other examples. The following relationships hold graphically:

$$A_1B_1 = d \qquad B_1C_1 = v \qquad = \left[1 - \frac{d^{(2)}}{2}\right]^2$$

$$A_2B_2 = \frac{d^{(2)}}{2} \qquad B_2C_2 = v^{1/2} \qquad = 1 - \frac{d^{(2)}}{2}$$

$$A_3B_3 = \frac{i^{(2)}}{2} \qquad B_3C_3 = (1+i)^{1/2} = 1 + \frac{i^{(2)}}{2}$$

$$A_4B_4 = i \qquad B_4C_4 = 1 + i = \left[1 + \frac{i^{(2)}}{2}\right]^2$$

It is interesting to note that nominal rates of interest and discount are not relevant under simple interest and simple discount. Since the amount of interest or discount is directly proportional to the time involved, a rate of interest of discount payable *m*thly is no different than one payable once per measurement period.

As with the word "discount," unfortunately the word "nominal" also has multiple meanings. In Section 7.3 "nominal" is used in another sense in connection with yields on bonds, while in Section 9.4 "nominal" takes on a different meaning in connection with the reflection of inflation in rates of interest.

Example 1.7 Find the accumulated value of $500 invested for five years at 8% per annum convertible quarterly.

The answer is

$$500\left[1 + \frac{.08}{4}\right]^{4\cdot 5} = 500(1.02)^{20}.$$

It should be noted that this situation is equivalent to one in which $500 is invested at a rate of interest of 2% for 20 years.

Example 1.8 Find the present value of $1000 to be paid at the end of six years at 6% per annum payable in advance and convertible semiannually.

The answer is

$$1000\left[1 - \frac{.06}{2}\right]^{2\cdot 6} = 1000(.97)^{12}.$$

It should be noted that this situation is equivalent to one in which the present value of $1000 to be paid at the end of 12 years is calculated at a rate of discount of 3%.

Example 1.9 Find the nominal rate of interest convertible quarterly which is equivalent to a nominal rate of discount of 6% per annum convertible monthly.

Using formula (1.22*a*)

$$\left[1 + \frac{i^{(4)}}{4}\right]^4 = \left[1 - \frac{.06}{12}\right]^{-12}$$

$$1 + \frac{i^{(4)}}{4} = (.995)^{-3}$$

$$i^{(4)} = 4[(.995)^{-3} - 1].$$

1.9 FORCES OF INTEREST AND DISCOUNT

The measures of interest defined in the preceding sections are useful for measuring interest over specified intervals of time. Effective rates of interest and discount measure interest over one measurement period, while nominal rates of interest and discount measure interest over $m$ths of a measurement period.

It is important in many cases to be able to measure the intensity with which interest is operating at each moment of time, i.e. over infinitesimally small intervals of time. This measure of interest at individual moments of time is called the *force of interest*.

Consider the investment of a fund such that the amount in the fund at time t is given by the amount function $A(t)$. Recall that the only factor operating on the fund is the growth of the fund through interest, i.e. no principal is added or withdrawn.

The intensity with which interest is operating at time t is measured by the rate of change or the slope of the $A(t)$ curve at time t. From elementary calculus, the slope of the $A(t)$ curve at time t is given by the derivative at that point.

However, as a measure of interest, $A'(t)$ is unsatisfactory, since it depends on the amount invested. For example, if \$200 and \$100 are invested under identical conditions, the rate of change of the \$200 fund will be twice as great as the rate of change of the \$100 fund. However, interest is not operating with twice the intensity on the \$200 fund; in fact, we would say that it is operating with the same intensity on both funds.

We can compensate for this by dividing $A'(t)$ by the amount in the fund at time t, namely $A(t)$. This gives a measure of the intensity with which interest is operating at time t expressed as a rate independent of the amount in the fund, i.e. as a rate per dollar in the fund. Thus, the force of interest at time t, denoted by δ_t, is defined as

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{a'(t)}{a(t)}. \tag{1.24}$$

The following properties of δ_t should be kept in mind:

1. δ_t is a measure of the intensity of interest at exact time t.
2. δ_t expresses this measurement as a rate per measurement period.

It is possible to write an expression for the value of $A(t)$ and $a(t)$ in terms of the function δ_t. It will be seen from formula (1.24) that an alternate expression for δ_t is

$$\delta_t = \frac{d}{dt} \log_e A(t) = \frac{d}{dt} \log_e a(t). \tag{1.25}$$

Replacing t by r and integrating both sides between the limits 0 and t

$$\int_0^t \delta_r dr = \int_0^t \frac{d}{dr} \log_e A(r) dr$$
$$= \log_e A(r)\Big]_0^t = \log_e \frac{A(t)}{A(0)}$$

and hence we have

$$e^{\int_0^t \delta_r dr} = \frac{A(t)}{A(0)} = \frac{a(t)}{a(0)} = a(t). \tag{1.26}$$

Another formula can be obtained from formula (1.24) written as $A(t)\delta_t = A'(t)$. Integrating between the limits 0 and n, we obtain

$$\int_0^n A(t)\,\delta_t\,dt = \int_0^n A'(t)\,dt = A(t)\Big]_0^n = A(n) - A(0). \tag{1.27}$$

Formula (1.27) has a rather interesting verbal interpretation. The term $A(n) - A(0)$ is the amount of interest earned over n measurement periods. The differential expression $A(t)\delta_t dt$ may be interpreted as the amount of interest earned on amount $A(t)$ at exact time t because of the force of interest δ_t. When this expression is integrated between limits 0 and n, it gives the total amount of interest earned over the n periods.

Further insight into the nature of the force of interest may be gained by analyzing formula (1.24) in terms of the definition of the derivative. The derivative of $A(t)$ may be expressed as

$$A'(t) = \lim_{h \to 0} \frac{A(t+h) - A(t)}{h}$$

and δ_t from formula (1.24) may then be written

$$\delta_t = \frac{A'(t)}{A(t)} = \lim_{h \to 0} \frac{A(t+h) - A(t)}{h A(t)}. \tag{1.28}$$

Now the expression $\dfrac{A(t+h) - A(t)}{hA(t)}$ may be regarded as the rate of interest based upon the interest earned during the interval from time t to time $t + h$. For example, if $h = 1$, we have $\dfrac{A(t+1) - A(t)}{A(t)}$, which is one period's increment in the fund divided by the amount in the fund at the beginning of the period. If $h = 1/2$, we have $2 \cdot \dfrac{A(t+1/2) - A(t)}{A(t)}$, which is twice half a period's increment in the fund divided by the amount in the fund at the beginning of the period. As h approaches 0, the limit of this expression, the force of interest, may be described as the nominal rate of interest based upon the intensity of interest at time t.

It is also possible to define a *force of discount* analogous to formula (1.24). For this purpose, we use the discount function $a^{-1}(t)$ instead of the accumulation function $a(t)$. The definition of the force of discount at time t, denoted by δ'_t, is given by

$$\delta'_t = - \frac{\frac{d}{dt} a^{-1}(t)}{a^{-1}(t)} . \tag{1.29}$$

The definition of δ'_t is completely analogous to the definition of δ_t except for the minus sign. The minus sign is necessary to make the force of discount a positive quantity. The denominator of formula (1.29) is positive but the numerator is negative, since $a^{-1}(t)$ is a decreasing function.

Later in this section it will be seen that the force of discount bears a relationship to nominal and effective rates of discount similar to the relationship that the force of interest bears to nominal and effective rates of interest. However, it can be shown that $\delta'_t = \delta_t$, so that we may dispense with δ'_t and use δ_t. The proof is as follows:

$$\begin{aligned}
\delta'_t &= - \frac{\frac{d}{dt} a^{-1}(t)}{a^{-1}(t)} \\
&= \frac{a^{-2}(t) \frac{d}{dt} a(t)}{a^{-1}(t)} \\
&= \frac{a^{-2}(t)\, a(t) \delta_t}{a^{-1}(t)} \quad \text{from formula (1.24)} \\
&= \delta_t .
\end{aligned}$$

In theory, the force of interest may vary instantaneously. However, in practice it is often a constant. If the force of interest is constant over an interval of time, then the effective rate of interest will also be constant over that interval. This can be seen by using formula (1.26) over n measurement periods (n a positive integer)

$$\begin{aligned}
e^{\int_0^n \delta_t \, dt} &= e^{n\delta} \text{ if } \delta_t = \delta \text{ for } 0 \leq t \leq n \\
&= a(n) \\
&= (1 + i)^n
\end{aligned}$$

so that

$$e^{\delta} = 1 + i \tag{1.30}$$

or

$$i = e^{\delta} - 1 \tag{1.31a}$$

which expresses i as a function of δ. Taking the logarithm of formula (1.30) expresses δ as a function of i

$$\delta = \log_e(1 + i). \tag{1.32a}$$

Formula (1.32*a*) may also be derived directly from the accumulation function for compound interest

$$\delta = \frac{\frac{d}{dt}(1+i)^t}{(1+i)^t} = \frac{(1+i)^t \log_e(1+i)}{(1+i)^t} = \log_e(1+i)$$

which is constant for all t.

Formulas (1.31*a*) and (1.32*a*) can also be written as series expansions:

$$i = e^{\delta} - 1 = \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \frac{\delta^4}{4!} + \cdots \tag{1.31b}$$

$$\delta = \log_e(1+i) = i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} + \cdots \tag{1.32b}$$

These two series will normally have a rapid rate of convergence, since in practice i and δ are usually small positive numbers whose successive powers diminish quickly.

Having related δ and i immediately relates δ and the other measures of interest described in this chapter. The following series of equalities is an expanded version of formula (1.22*a*) which summarizes much of the material contained in this chapter:

$$\left[1 + \frac{i^{(m)}}{m}\right]^m = 1 + i = v^{-1} = (1-d)^{-1} = \left[1 - \frac{d^{(p)}}{p}\right]^{-p} = e^{\delta}. \tag{1.33}$$

It is instructive to examine the behavior of the force of interest under simple interest and simple discount. For simple interest we have

$$\begin{aligned}\delta_t &= \frac{\frac{d}{dt}a(t)}{a(t)} \\ &= \frac{\frac{d}{dt}(1+it)}{1+it} \\ &= \frac{i}{1+it} \quad \text{for } 0 \le t.\end{aligned} \tag{1.34}$$

Similarly, for simple discount we have

$$\delta_t = \delta'_t = -\frac{\frac{d}{dt}a^{-1}(t)}{a^{-1}(t)}$$

$$= -\frac{\frac{d}{dt}(1-dt)}{1-dt}$$

$$= \frac{d}{1-dt} \text{ for } 0 \leq t < 1/d. \tag{1.35}$$

The upper restriction on values of t in formula (1.35) is necessary to have values of δ_t that are finite and positive. As we would expect, δ_t is a decreasing function of t for simple interest, but an increasing function of t for simple discount.

The reader may be surprised to observe that although a constant force of interest δ leads to a constant effective rate of interest i, the reverse is not necessarily true. To see this possibility, consider n measurement periods (n a positive integer). Again, we have

$$a(n) = (1+i)^n = e^{\int_0^n \delta_r dr}.$$

However, if we subdivide the n-period integral into a sum of one-period integrals, we have

$$(1+i)^n = e^{\left[\int_0^1 \delta_t dt + \int_1^2 \delta_t dt + \cdots + \int_{n-1}^n \delta_t dt\right]}$$

$$= e^{\int_0^1 \delta_t dt} e^{\int_1^2 \delta_t dt} \cdots e^{\int_{n-1}^n \delta_t dt}.$$

If we allow δ_t to vary within each of these measurement periods in such a way that all the one-period integrals are equal, then we would have the result that i is constant for each of the n periods, but δ_t varies within each of these periods.

A good practical example to illustrate this phenomenon is to invest 1 for n periods at compound interest for completed periods, but use simple interest over fractional periods. This produces a growth pattern of $(1+i)^k$ for $k = 1, 2, \ldots n$, so that the effective rate of interest is constant over the n measurement periods. However, δ_t varies within each of these periods, since δ_t varies under simple interest. Other examples could be constructed to illustrate the same point, but they would generally involve more esoteric patterns for the variation in δ_t.

Another interesting insight into the nature of the force of interest can be obtained by analyzing δ in terms of $i^{(m)}$. From formula (1.33)

$$\left[1 + \frac{i^{(m)}}{m}\right]^m = e^{\delta}$$

$$i^{(m)} = m\left[e^{\frac{\delta}{m}} - 1\right].$$

Now using a series expansion, we have

$$i^{(m)} = m\left[\frac{\delta}{m} + \frac{1}{2!}\left(\frac{\delta}{m}\right)^2 + \frac{1}{3!}\left(\frac{\delta}{m}\right)^3 + \cdots\right]$$

$$= \delta + \frac{\delta^2}{2!m} + \frac{\delta^3}{3!m^2} + \cdots$$

and taking the limit as m approaches infinity,

$$\lim_{m\to\infty} i^{(m)} = \delta. \tag{1.36}$$

This formula has intuitive appeal. Since $i^{(m)}$ is a nominal rate of interest convertible $m$thly, we can interpret δ as a nominal rate of interest convertible continuously.

By an analogous argument, it is possible to show that

$$\lim_{m\to\infty} d^{(m)} = \delta. \tag{1.37}$$

The proof is left as an exercise. This formula also has intuitive appeal. Since $d^{(m)}$ is a nominal rate of discount convertible $m$thly, we can interpret δ as a nominal rate of discount convertible continuously. In essence, this is an alternate proof that the force of interest and the force of discount are equal. A third proof of this result is left as an exercise.

The force of interest is a useful conceptual device, making the continuous growth of money at compound interest similar to growth functions encountered in the natural sciences. In theory, the most fundamental measure of interest is the force of interest. In practice, however, effective and nominal rates of interest and discount tend to be used more frequently because they are simpler for most people to comprehend and because most financial transactions involve discrete, not continuous, processes. This does not mean that the force of interest is devoid of practical significance. Besides being a useful conceptual and analytical tool, it can be used in practice as an approximation to interest converted very frequently, such as daily. Also, some financial institutions actually have started to use continuous compounding in recent years.

***Example 1.10** Find the accumulated value of \$1000 invested for ten years if the force of interest is 5%.*

The answer is

$$1000e^{(.05)(10)} = 1000e^{.5}.$$

1.10 VARYING INTEREST

This section is concerned with situations involving varying interest. Two types of variation are considered. Other types of variation can be analyzed from basic principles.

The first type of variation considered is a continuously varying force of interest. The basic formula for use in problems involving a varying force of interest is formula (1.26)

$$a(t) = e^{\int_0^t \delta_r dr}. \tag{1.26}$$

If the form of δ_r is readily integrable, results may be obtained directly. If the form of δ_r is not readily integrable, approximate methods of integration are necessary.

The second type of variation considered involves changes in the effective rate of interest over a period of time. This type of variation is probably the one most commonly encountered in practice. As before, let i_n denote the effective rate of interest during the nth period from the date of investment. Then for integral $t \geq 1$, we have

$$a(t) = (1 + i_1)(1 + i_2) \cdots (1 + i_t) = \prod_{k=1}^{t} (1 + i_k). \tag{1.38}$$

If $i_1 = i_2 = \ldots = i_t = i$, the familiar result $a(t) = (1 + i)^t$ is obtained.

Present values with varying effective rates of interest can be handled similarly. Again, for integral t $\geq$ 1 we have

$$a^{-1}(t) = (1 + i_1)^{-1}(1 + i_2)^{-1} \cdots (1 + i_t)^{-1} = \prod_{k=1}^{t} (1 + i_k)^{-1} = \prod_{k=1}^{t} v_k. \tag{1.39}$$

The approach followed in formulas (1.38) and (1.39) can be easily generalized to include nominal rates of interest or discount. All that is necessary is to count the number of interest conversion periods in each bracketed subinterval in which the rates are constant and then multiply the appropriate factors for each subinterval across the entire interval. This procedure is illustrated in the exercises.

Frequently, in situations involving varying interest it is desired to find an equivalent level rate to the rates that vary. When this situation is encountered, the definition of equivalency given in Section 1.7 can be applied directly. However, it is important to note that the answers will depend on the period of time chosen for the comparison. Since interest is varying, the rate that would be equivalent over a period of one length would not be the same as that over a period of a different length.

Example 1.11 ***Find the accumulated value of 1 at the end of n years if***

$$\delta_t = \frac{1}{1+t}.$$

Using formula (1.26), the answer is

$$e^{\int_0^n \delta_t\,dt} = e^{\int_0^n \frac{1}{(1+t)}\,dt} = e^{\log_e(1+t)]_0^n} = 1 + n.$$

Example 1.12 ***Find the accumulated value of \$1000 at the end of 15 years if the effective rate of interest is 5% for the first 5 years, 4 1/2% for the second 5 years, and 4% for the third 5 years.***

Using formula (1.38), the answer is

$$1000(1.05)^5(1.045)^5(1.04)^5.$$

1.11 SUMMARY OF RESULTS

Table 1.1 summarizes much of the material of this chapter.

Table 1.1 Summary of Relationships in Chapter 1

Rate of interest or discount	*The accumulated value of 1 at time* $t = a(t)$	*The present value of 1 at time* $t = a^{-1}(t)$
Compound interest		
i	$(1 + i)^t$	$v^t = (1 + i)^{-t}$
$i^{(m)}$	$\left(1 + \frac{i^{(m)}}{m}\right)^{mt}$	$\left(1 + \frac{i^{(m)}}{m}\right)^{-mt}$
d	$(1 - d)^{-t}$	$(1 - d)^t$
$d^{(m)}$	$\left(1 - \frac{d^{(m)}}{m}\right)^{-mt}$	$\left(1 - \frac{d^{(m)}}{m}\right)^{mt}$
δ	$e^{\delta t}$	$e^{-\delta t}$
Simple interest		
i	$1 + it$	$(1 + it)^{-1}$
Simple discount		
d	$(1 - dt)^{-1}$	$1 - dt$

EXERCISES

1.2 The accumulation and amount functions

Consider the amount function $A(t) = t^2 + 2t + 3$.

a) Find the corresponding accumulation function $a(t)$.

b) Verify that $a(t)$ satisfies the three properties of an accumulation function.

c) Find I_n.

2. *a*) Prove that $A(n) - A(0) = I_1 + I_2 + \cdots + I_n$.
   b) Verbally interpret the result obtained in (*a*).

3. Find the amount of interest earned between time t and time n, where $t < n$, if:
   a) $I_r = r$.
   b) $I_r = 2^r$.

4. It is known that $a(t)$ is of the form $at^2 + b$. If \$100 invested at time 0 accumulates to \$172 at time 3, find the accumulated value at time 10 of \$100 invested at time 5.

1.3 The effective rate of interest

5. Assume that $A(t) = 100 + 5t$.
   a) Find i_5.
   b) Find i_{10}.

6. Assume that $A(t) = 100(1.1)^t$.
   a) Find i_5.
   b) Find i_{10}.

7. Show that $A(n) = (1 + i_n)A(n - 1)$.

8. If $A(4) = 1000$ and $i_n = .01n$, find $A(7)$.

1.4 Simple interest

9. *a*) At what rate of simple interest will \$500 accumulate to \$615 in 2 1/2 years?
   b) In how many years will \$500 accumulate to \$630 at 7.8% simple interest?

10. If i_k is the rate of simple interest for period k, where $k = 1, 2, \ldots, n$, show that $a(n) - a(0) = i_1 + i_2 + \cdots + i_n$.

11. At a certain rate of simple interest \$1000 will accumulate to \$1110 after a certain period of time. Find the accumulated value of \$500 at a rate of simple interest three fourths as great over twice as long a period of time.

12. Simple interest of $i = 4\%$ is being credited to a fund. In which period is this equivalent to an effective rate of 2 1/2%?

1.5 Compound interest

13. Assuming that $0 < i < 1$, show that:
   a) $(1 + i)^t < 1 + it$ if $0 < t < 1$.
   b) $(1 + i)^t = 1 + it$ if $t = 1$.
   c) $(1 + i)^t > 1 + it$ if $t > 1$.
   This exercise verifies the relative magnitudes of accumulated values at simple and compound interest over various periods of time.

14. It is known that \$600 invested for two years will earn \$264 in interest. Find the accumulated value of \$2000 invested at the same rate of compound interest for three years.

15. Show that the ratio of the accumulated value of 1 invested at rate i for n periods, to the accumulated value of 1 invested at rate j for n periods, $i > j$, is equal to the accumulated value of 1 invested for n periods at rate r. Find an expression for r as a function of i and j.

16. At a certain rate of compound interest, 1 will increase to 2 in a years, 2 will increase to 3 in b years, and 3 will increase to 15 in c years. If 6 will increase to 10 in n years, express n as a function of a, b, and c.

17. An amount of money is invested for one year at a rate of interest of 3% per quarter. Let $D(k)$ be the difference between the amount of interest earned on a compound interest basis and on a simple interest basis for quarter k, where $k = 1, 2, 3, 4$. Find the ratio of $D(4)$ to $D(3)$.

1.6 Present value

18. Find an expression for the discount factor during the nth period from the date of investment, i.e. $(1 + i_n)^{-1}$ in terms of the amount function.

19. The sum of the present value of 1 paid at the end of n periods and 1 paid at the end of $2n$ periods is 1. Find $(1 + i)^{2n}$.

20. Show that the current value of a payment of 1 made n periods ago and a payment of 1 to be made n periods in the future is greater than 2, if $i > 0$.

21. It is known that an investment of \$500 will increase to \$4000 at the end of 30 years. Find the sum of the present values of three payments of \$10,000 each which will occur at the end of 20, 40, and 60 years.

1.7 The effective rate of discount

22. The amount of interest earned on A for one year is \$336, while the equivalent amount of discount is \$300. Find A.

23. *a*) Find d_5 if the rate of simple interest is 10%.
b) Find d_5 if the rate of simple discount is 10%.

24. *a*) Assuming compound discount, show that d_n is constant for all n.
b) Assuming simple discount, show that d_n is increasing for increasing n if $0 < n - 1 < 1/d$.

25. Assuming that $0 < d < 1$, show that:
a) $(1 - d)^t < 1 - dt$ if $0 < t < 1$.
b) $(1 - d)^t = 1 - dt$ if $t = 1$.
c) $(1 - d)^t > 1 - dt$ if $t > 1$.
This exercise verifies the relative magnitudes of present values at simple and compound discount over various periods of time.

26. If i and d are equivalent rates of simple interest and simple discount over t periods, show that
$$i - d = idt.$$

27. Show that
$$\frac{d^3}{(1-d)^2} = \frac{(i-d)^2}{1-v}.$$

1.8 Nominal rates of interest and discount

28. *a*) Express $d^{(4)}$ as a function of $i^{(3)}$.
b) Express $i^{(6)}$ as a function of $d^{(2)}$.

29. On occasion, interest is convertible less frequently than once a year. Define $i^{(1/m)}$ and $d^{(1/m)}$ to be nominal annual rates of interest and discount convertible once every m years. Find a formula analogous to formula (1.22*a*) for this situation.

30. Find the accumulated value of \$100 at the end of two years:
a) If the nominal annual rate of interest is 6% convertible quarterly.
b) If the nominal annual rate of discount is 6% convertible once every four years.

31. Derive formula (1.23).

32. *a*) Show that $i^{(m)} = d^{(m)}(1+i)^{1/m}$.
b) Verbally interpret the result obtained in (*a*).

33. Given that $i^{(m)} = .1844144$ and $d^{(m)} = .1802608$, find m.

34. It is known that
$$1 + \frac{i^{(n)}}{n} = \frac{1 + \frac{i^{(4)}}{4}}{1 + \frac{i^{(5)}}{5}}.$$
Find n.

35. If $r = \dfrac{i^{(4)}}{d^{(4)}}$, express v in terms of r.

1.9 Forces of interest and discount

36. Derive formula (1.37).

37. Use formula (1.23) to give a third proof of the result that $\delta' = \delta$.

38. Rank $i, i^{(m)}, d, d^{(m)}$, and δ in increasing order of magnitude, assuming $m > 1$.

39. Show that $\dfrac{d}{dt}\delta_t = \dfrac{A''(t)}{A(t)} - \delta_t^2$.

40. *a*) Obtain an expression for δ_t if $A(t) = Ka^t b^{t^2} d^{c^t}$.
b) Is formula (1.24) or (1.25) more convenient in this case?

41. Show that:

    a) $\int_0^n \delta_t dt = -\log_e v^n$.

    b) $\int_0^n A(t)\delta_t dt = I_1 + I_2 + \cdots + I_n$.

42. Fund A accumulates at a simple interest rate of 10%. Fund B accumulates at a simple discount rate of 5%. Find the point in time at which the forces of interest on the two funds are equal.

43. An investment is made for one year in a fund whose accumulation function is a second degree polynomial. The nominal rate of interest earned during the first half of the year is 5% convertible semiannually. The effective rate of interest earned for the entire year is 7%. Find $\delta_{.5}$.

44. Find an expression for the fraction of a period at which the excess of accumulated values computed at simple interest over compound interest is a maximum.

1.10 Varying interest

45. If $\delta_t = .01t, 0 \le t \le 2$, find the equivalent annual effective rate of interest over the interval $0 \le t \le 2$.

46. Find the accumulated value of 1 at the end of 19 years if $\delta_t = .04(1 + t)^{-2}$.

47. Find the level effective rate of interest over a three-year period which is equivalent to an effective rate of discount of 8% the first year, 7% the second year, and 6% the third year.

48. *a)* Find the accumulated value of 1 at the end of n periods where the effective rate of interest for the kth period, $1 \le k \le n$, is defined by

$$i_k = (1 + r)^k(1 + i) - 1.$$

    b) Show that the answer to (*a*) can be written in the form $(1 + j)^n$. Find j.

49. The force of interest at time t is $t^3/100$. Find $a^{-1}(3)$.

50. In Fund X money accumulates at a force of interest

$$\delta_t = .01t + .1 \text{ for } 0 \le t \le 20.$$

    In Fund Y money accumulates at an annual effective interest rate i. An amount of 1 is invested in each fund for 20 years. The value of Fund X at the end of 20 years is equal to the value of Fund Y at the end of 20 years. Calculate the value of Fund Y at the end of 1.5 years.

51. You are given $\delta_t = \dfrac{2}{t - 1}$ for $2 \le t \le 10$. For any one year interval between n and $n + 1$, with $2 \le n \le 9$, calculate the equivalent $d^{(2)}$.

52. If the effective rate of discount in year k is equal to $.01k + .06$ for $k = 1, 2, 3$, find the equivalent rate of simple interest over the three-year period.

Miscellaneous problems

53. *a)* Show that $\delta \doteq \dfrac{i^{(m)} + d^{(m)}}{2}$, where $\doteq$ denotes approximate equality.

    b) Assuming $m = 1$, find an exact expression for the error in (*a*) expressed as a series expansion in δ.

54. Show that
$$\sum_{m=1}^{\infty} (-1)^{m-1} i^m \left[\frac{1}{d^{(m)}} - \frac{1}{i^{(m)}} \right] = \delta.$$

55. Find the following derivatives:

    a) $\dfrac{d}{di} d$ *c)* $\dfrac{d}{dd} d^{(m)}$ *e)* $\dfrac{d}{d\delta} d$

    b) $\dfrac{d}{di} \delta$ *d)* $\dfrac{d}{dv} \delta$

56. Find the following in the form of series expansions:

    a) i as a function of d. *d)* $i^{(m)}$ as a function of i.

    b) d as a function of i. *e)* δ as a function of d.

    c) v as a function of δ.

57. Show that
$$\delta = \frac{d + i}{2} + \frac{d^2 - i^2}{4} + \frac{d^3 + i^3}{6} + \cdots$$

58. Show that
$$\frac{d^n}{dv^n}(v^{n-1}\delta) = -(1 + i)(n - 1)!$$

59. *a)* (1) Derive an expression for $a(t)$ assuming δ_r is linear and positive, i.e. $\delta_r = a + br$, where $a > 0$ and $b > 0$.

    (2) Find the accumulation factor during the nth period from the date of investment, i.e. $1 + i_n$.

    b) (1) Derive an expression for $a(t)$ assuming δ_r is exponential and positive, i.e. $\delta_r = ab^r$, where $a > 0$ and $b > 0$.

    (2) Find the accumulation factor during the nth period from the date of investment, i.e. $1 + i_n$.

60. *Stoodley's formula* for the force of interest is
$$\delta_t = p + \frac{s}{1 + re^{st}}.$$
Show that
$$a^{-1}(t) = \frac{1}{1 + r} v_1^t + \frac{r}{1 + r} v_2^t$$
where $v_1 = e^{-(p+s)}$ and $v_2 = e^{-p}$.

2

Solution of problems in interest

2.1 INTRODUCTION

The basic principles in the theory of interest are relatively few. In Chapter 1, various quantitative measures of interest were analyzed. Chapter 2 discusses general principles to be followed in the solution of problems in interest. The purpose of this chapter is to develop a systematic approach by which the basic principles from Chapter 1 can be applied to more complex financial transactions.

With a thorough understanding of the first two chapters it is possible to solve most problems in interest. Successive chapters have two main purposes:

1. To familiarize the reader with more complex types of financial transactions, including definitions of terms, which occur in practice.
2. To provide a systematic analysis of these financial transactions, which will often lead to a more efficient handling of the problem than resorting to basic principles.

As a result of the second purpose above, on occasion, simplifying formulas will be derived. Fortunately, the number of formulas requiring memorization is small. Even so, a common source of difficulty for some is blind reliance on formulas without an understanding of the basic principles upon which the formulas are based. It is important to realize that problems in interest can generally be solved from basic principles and that in many cases resorting to basic principles is not as inefficient as it may first appear to be.

2.2 OBTAINING NUMERICAL RESULTS

In Chapter 1 answers to many of the examples and exercises involving accumulated values and present values and involving equivalent rates of interest

and discount were left in exponential form. This was done to facilitate the presentation of the basic principles without the necessity of the reader performing an undue amount of arithmetical computation. Naturally, in practical work, actual numerical answers are usually desired, and the purpose of this section is to discuss the various possible methods of obtaining such answers.

The advent of personal computers and inexpensive pocket calculators with exponential and logarithmic functions makes it feasible to obtain numerical answers to problems in interest by direct calculation. In fact, direct calculation is probably the easiest and most efficient method in most cases.

An alternative approach is to use compound interest tables. Such tables appear in Appendix I and include values of v^n and $(1+i)^n$, as well as other functions which will be defined in succeeding chapters, for several rates of interest and for values of n from 1 to 50. Also, nonintegral values are given for v^k and $(1+i)^k$ for $k = 1/2$, $1/4$, and $1/12$. Use of the compound interest tables is a convenient approach if required values appear in the tables.

As a last resort, if neither of the above techniques can be applied, direct calculation by hand can still be used. This may require the use of series expansions. One example would be to evaluate $(1+i)^k$ using the binomial theorem

$$(1+i)^k = 1 + ki + \frac{k(k-1)}{2!}i^2 + \frac{k(k-1)(k-2)}{3!}i^3 + \cdots \tag{2.1}$$

A second example would be to evaluate $e^{k\delta}$ as

$$e^{k\delta} = 1 + k\delta + \frac{(k\delta)^2}{2!} + \frac{(k\delta)^3}{3!} + \cdots \tag{2.2}$$

These formulas can be used for either positive or negative values of k, although convergence may be slow unless the absolute value of k is small (e.g. for fractional periods of time). Other series expansions can be developed from the basic identities in Chapter 1. However, again it should be emphasized that using series expansions for calculation purposes is cumbersome and should be unnecessary except in unusual circumstances.

One method of crediting interest that is commonly encountered in practice is to use compound interest for integral periods of time, but to use simple interest for any fractional period. This approach is equivalent to using only the first two terms of the binomial expansion in formula (2.1) assuming $0 < k < 1$.

It can be shown that the use of simple interest for a final fractional period is equivalent to performing a linear interpolation between $(1+i)^n$ and

$(1 + i)^{n+1}$, where n is a non-negative integer. To see this we start with the linear interpolation

$$(1 + i)^{n+k} \doteq (1 - k)(1 + i)^n + k(1 + i)^{n+1} \tag{2.3}$$

$$= (1 + i)^n[(1 - k) + k(1 + i)]$$

$$= (1 + i)^n(1 + ki) \tag{2.4}$$

which is the formula if simple interest is used over the final fractional period. The use of simple interest introduces a bias, since it was shown in Section 1.5 that simple interest produces a larger accumulated value over fractional periods than does compound interest.

It is interesting to note that, in an analogous fashion, linear interpolation for v^{n+k} is equivalent to using simple discount over the final fractional period. Again, starting with linear interpolation

$$v^{n+k} = (1 - d)^{n+k} \doteq (1 - k)(1 - d)^n + k(1 - d)^{n+1} \tag{2.5}$$

$$= (1 - d)^n[(1 - k) + k(1 - d)]$$

$$= v^n(1 - kd) \tag{2.6}$$

which is the formula if simple discount is used over the final fractional period.

Example 2.1 Obtain numerical answers for Examples 1.7, 1.8, 1.9, 1.10, and 1.12.

1. *Example 1.7*:
   From the interest tables or by direct calculation
   $$500(1.02)^{20} = 500(1.48595) = \$742.97.$$

2. *Example 1.8*:
   By direct calculation
   $$1000(.97)^{12} = 1000(.69384) = \$693.84.$$

3. *Example 1.9*:
   By direct calculation
   $$i^{(4)} = 4[(.995)^{-3} - 1] = .0606, \text{ or } 6.06\%.$$

4. *Example 1.10*:
   By direct calculation
   $$1000e^{.5} = \$1648.72.$$
   If necessary, a series expansion could be used,
   $$1000\left[1 + .5 + \frac{(.5)^2}{2!} + \frac{(.5)^3}{3!} + \frac{(.5)^4}{4!} + \frac{(.5)^5}{5!} + \frac{(.5)^6}{6!} + \cdots\right] = \$1648.72.$$

Terms to the sixth power must be used to achieve the same level of accuracy, which illustrates why series expansions should be used to obtain numerical answers only as a last resort.

5. *Example 1.12:*
From the interest tables, or by direct calculation

$$1000(1.05)^5(1.045)^5(1.04)^5 = 1000(1.27628)(1.24618)(1.21665) = \$1935.05.$$

Example 2.2 Find the accumulated value of $5000 at the end of 30 years and 4 months at 6% per annum convertible semiannually: (1) assuming compound interest throughout, and (2) assuming simple interest during the final fractional period.

1. Assuming compound interest throughout, the answer is

$$5000(1.03)^{60\,2/3} = \$30,044.27$$

by direct calculation.

2. Assuming simple interest during the final fractional period, the answer is

$$5000(1.03)^{60}(1.02) = \$30,047.18.$$

The answer to 2 is larger than 1, illustrating that simple interest produces larger accumulated values over fractional periods than compound interest does, although the difference is quite small.

2.3 DETERMINING TIME PERIODS

In practical problems involving interest it is necessary to determine the time period of an investment. Although there would appear to be no ambiguity in this process, different methods of counting the days in a period of investment have arisen in practice. Three methods are commonly encountered.

The first method is to use the exact number of days for the period of investment and to use 365 days in a year. Simple interest computed on this basis is sometimes called *exact simple interest* and is often denoted by "actual/actual." Appendix II contains a table numbering the days of the year, which facilitates counting the number of days between two given dates.

The second method assumes that each calendar month has 30 days and that the entire calendar year has 360 days. Simple interest computed on this basis is sometimes called *ordinary simple interest* and is often denoted by "30/360." Appendix II cannot be used for calculations on this basis. However, a formula for computing the number of days between two given dates is

$$360(Y_2 - Y_1) + 30(M_2 - M_1) + (D_2 - D_1) \tag{2.7}$$

where M_1 = month of first date
D_1 = day of first date
Y_1 = year of first date
M_2 = month of second date
D_2 = day of second date
Y_2 = year of second date.

The third method is a hybrid and uses the exact number of days for the period of investment, but uses 360 days in a year. Simple interest on this basis is sometimes called the *Banker's Rule* and is often denoted by "actual/360." It will be shown in the exercises that the Banker's Rule is always more favorable to a lender than is exact simple interest. Also, it will be shown that the Banker's Rule is usually more favorable to a lender than is ordinary simple interest, although exceptions do exist.

In theory we could have a fourth method denoted by "30/actual" or "30/365." However, this method is almost never encountered in practice.

A further complication arises in a leap year. In most cases, February 29 is counted as a day and the year has 366 days. However, in some cases, February 29 is counted as a day, but the year is still counted as having 365 days. In other cases, February 29 is not counted as day, i.e. no interest is earned. The author has even encountered the assumption that all years have 365 1/4 days! A uniform approach to leap year does not seem to have emerged and different calculation bases are encountered in practice. It should be noted that under ordinary simple interest (30/360), leap year is irrelevant.

The above terms and discussion have been couched in terms of simple interest. However, the three commonly encountered calculation bases, i.e. actual/actual, 30/360, and actual/360, also are used for calculations on a compound interest basis.

It is assumed, unless stated otherwise, that in counting days interest is not credited for both the date of deposit and the date of withdrawal, but for only one of these two dates. If the difference in the two dates is calculated by normal procedures, then this will be the result. However, situations are occasionally encountered in practice in which interest is credited for both the date of deposit and the date of withdrawal, resulting in one extra day's interest.

Not all practical problems involving interest require the counting of days. Many financial transactions are handled on a monthly, quarterly, semiannual, or annual basis. In these cases the counting methods described in this section are not required.

Example 2.3 ***Find the amount of interest that $2000 deposited on June 17 will earn, if the money is withdrawn on September 10 in the same year and if the rate of interest is 8%, on the following bases: (1) exact simple interest (actual/actual), (2) ordinary simple interest (30/360), and (3) the Banker's Rule (actual/360).***

1. From Appendix II, September 10 is day 253 and June 17 is day 168. The actual number of days in the period of investment is 253 - 168 = 85. Thus, the answer is
$$2000(.08)\left[\frac{85}{365}\right] = \$37.26$$
assuming that the year in question is not a leap year. Note that even if a table such as Appendix II is unavailable, it is an easy matter to count the number of days and arrive at 85.
2. Using formula (2.7), the number of days is
$$360(0) + 30(9 - 6) + (10 - 17) = 83.$$
Thus, the answer is
$$2000(.08)\left[\frac{83}{360}\right] = \$36.89.$$
3. The counting has already been done above and the answer is
$$2000(.08)\left[\frac{85}{360}\right] = \$37.78.$$
Not surprisingly, the answer using the Banker's Rule is greater than using either exact simple interest or ordinary simple interest.

2.4 THE BASIC PROBLEM

Broken down into its simplest terms, an interest problem involves four basic quantities:

1. The principal originally invested.
2. The length of the investment period.
3. The rate of interest.
4. The accumulated value of the principal at the end of the investment period.

If any three of these quantities are known, then the fourth quantity can be determined. In the problems involving accumulated values considered so far, No. 4 is the unknown quantity; whereas, in the problems involving present values, No. 1 is the unknown quantity. Section 2.6 considers the case in which No. 2 is the unknown, while Section 2.7 considers the case in which No. 3 is the unknown.

The following observations may prove helpful in the solution of problems in interest:

a) The reader should assess the tools that will be available in performing the financial calculations. For example, the following considerations are relevant for the tools commonly utilized:

Interest tables

- What interest rates are available?
- Which functions are provided?
- How many time periods have been calculated?
- Are enough decimal places carried for sufficient accuracy?

Pocket calculators

- Does the calculator have exponential and logarithmic functions?
- Is it programmable?
- Does it have built-in algorithms for financial calculations?

Personal computers

- Is software already available to do the calculations involved?
- Is the volume of calculations sufficiently large to warrant use of a personal computer?
- Is the ability to quickly generate printouts of results important?

The approach to be used will depend on the tools available, the type and complexity of the calculations involved, and the volume of computation involved. For the balance of this book we will assume the availability of a pocket calculator with exponential and logarithmic functions and we will utilize the interest tables in Appendix I. We will not assume that a personal computer is available; however, many applications can readily be handled by commonly available software or can readily be programmed.

b) The length of the investment period is measured in time units. It was mentioned in Chapter 1 that the fundamental measurement period is often assumed to be one year, and many problems are worked with this as the time unit, especially those involving effective rates of interest or discount.

However, if nominal rates of interest or discount are involved, often a time unit other than one year is most advantageous. The most convenient time unit to use is generally the interest conversion period. In problems involving continuous interest, however, some other time unit, such as a year, must typically be used.

c) An interest problem can be viewed from two perspectives, since it involves a financial transaction between two parties, the borrower and the lender. From either perspective, the problem is essentially the same; however, the

wording of a problem may be different depending upon the point of view. Examples and exercises phrased from both points of view appear, and the reader should not let the different phraseology be a source of confusion.

As an example, recall the discussion in Section 1.7 involving the use of the words "paid" or "credited." To some readers the word "paid" may seem more normal from the vantage point of the borrower, while "credited" may seem more normal from the vantage point of the lender. Many other such examples could be cited.

Complex financial transactions often involve more than two parties. For example, a business firm analyzing its rate of return on a major investment in a new plant is involved with a multiplicity of parties. However, the basic principles developed to analyze two-party transactions can readily be extended to analyze these more complex transactions.

d) In practical applications involving interest the terminology can become confusing. Many terms have ambiguous meanings (e.g. see the discussion on the use of the word "discount" in Section 1.7). Furthermore, as we shall see in succeeding chapters some terms used unfortunately do not convey an intuitive description of the transactions involved (e.g. see the discussion on the terms "annuity-immediate" and "annuity-due" in Section 3.3). Finally, many parties involved in financial transactions simply do not always use terms with the precise meanings and definitions contained in this book. The reader is admonished in real-world applications to look beyond the stated terms and be certain to understand the exact nature of the financial transactions in question. There simply is not total consistency in terminology among the large and diverse number of parties involved in financial transactions involving interest.

2.5 EQUATIONS OF VALUE

It is a fundamental principle in the theory of interest that the value of an amount of money at any given point in time depends upon the time elapsed since the money was paid in the past or upon the time which will elapse in the future before it is paid. We have already seen this in many of the examples and exercises considered thus far in the first two chapters.

This principle is often characterized as the recognition of the *time value of money*. This process would be in contrast to financial calculations not involving the effect of interest, in which case it would be said that such calculations do not recognize the time value of money. The reader is cautioned that "recognition of the time value of money" reflects the effect of interest, but not

the effect of inflation which reduces the purchasing power of money over time. Inflation-adjusted calculations will be discussed in Section 9.4.

As a consequence of the above principle, it is obvious that two or more amounts of money payable at different points in time cannot be compared until all the amounts are accumulated or discounted to a common date. This common date is called the *comparison date*, and the equation which accumulates or discounts each payment to the comparison date is called the *equation of value*.

One device which is often helpful in the solution of equations of value is the *time diagram*. A time diagram is a one-dimensional diagram in which units of time are measured along the one dimension and payments are placed on the diagram at the appropriate points. Note that payments in one direction are placed on the top of the diagram and payments in the other direction are placed on the bottom of the diagram. The comparison date is denoted by an arrow. Figure 2.1 is an example of a time diagram used in the solution of Example 2.4.

The time diagram is not necessary in the solution of equations of value; it is merely an aid in visualizing the problem. With some practice the reader can usually dispense with a time diagram on simpler problems. However, time diagrams are usually helpful in the solution of more complex problems. One of the properties of compound interest is that the choice of the comparison date makes no difference in the answer obtained. Thus, there is a different equation of value for each comparison date, but they all produce the same answer. This important property of compound interest will be illustrated in the solution of Example 2.4.

The reader is cautioned that under other patterns of interest, e.g. simple interest or simple discount, the choice of a comparison date does affect the answer obtained. This illustrates once again the inherent inconsistency in using simple interest or simple discount.

The reader should be aware that the problems involving accumulated values and present values already considered in the first two chapters are examples of equations of value. Example 2.4 illustrates a more general type of problem.

***Example 2.4** In return for a promise to receive $600 at the end of 8 years, a person agrees to pay $100 at once, $200 at the end of 5 years, and to make a further payment at the end of 10 years. Find the payment at the end of 10 years if the nominal rate of interest is 8% convertible semiannually.*

We shall first work the problem with a comparison date of the present. The time diagram is shown in Figure 2.1.

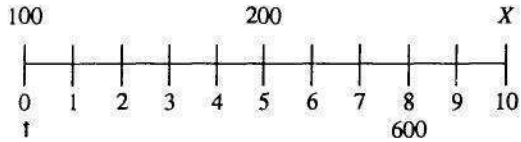

Figure 2.1 Time Diagram for Example 2.4

Since interest is convertible semiannually, we will count time periods in half-years. The equation of value is

$$100 + 200v^{10} + Xv^{20} = 600v^{16} \text{ at } 4\%$$

$$X = \frac{600v^{16} - 100 - 200v^{10}}{v^{20}}$$

$$= \frac{600(.53391) - 100 - 200(.67556)}{.45639}$$

$$= \$186.76.$$

We could also have chosen a different comparison date and obtained a different equation of value. For example, if the comparison date were chosen to be the end of the 10th year, then the arrow in the time diagram would be under 10 and the equation of value would be

$$100(1.04)^{20} + 200(1.04)^{10} + X = 600(1.04)^4$$

$$X = 600(1.04)^4 - 100(1.04)^{20} - 200(1.04)^{10}$$

$$= 600(1.16986) - 100(2.19112) - 200(1.48024)$$

$$= \$186.76.$$

Thus, the same answer is obtained. The two equations of value are equivalent. If both sides of the first one are multiplied by $(1.04)^{20}$, the second one is obtained. The reader can verify that if other comparison dates are chosen, the same answer is obtained.

2.6 UNKNOWN TIME

As discussed in Section 2.4, if any three of the four basic quantities entering into an interest problem are given, then the fourth can be determined. In this section we consider the situation in which the length of the investment period is the unknown.

The best method of solving for unknown time involving a single payment is to use logarithms. This technique will be illustrated in Example 2.5. As indicated above in Section 2.4, we are assuming the availability of a pocket calculator with exponential and logarithmic functions.

An alternative approach with less accuracy that can be used if such a calculator is unavailable is linear interpolation in the interest tables. This technique is also illustrated in Example 2.5.

A situation sometimes arises in which several payments made at various points in time are to be replaced by one payment numerically equal to the sum

of the other payments. The problem is to find the point in time at which the one payment should be made such that it is equivalent in value to the payments made separately.

Let amounts $s_1, s_2, \ldots, s_n$ be paid at times $t_1, t_2, \ldots, t_n$ respectively. The problem is to find time t, such that $s_1 + s_2 + \cdots + s_n$ paid at time t is equivalent to the payments of $s_1, s_2, \ldots, s_n$ made separately.

The fundamental equation of value is

$$(s_1 + s_2 + \cdots + s_n)v^t = s_1v^{t_1} + s_2v^{t_2} + \cdots + s_nv^{t_n} \tag{2.8}$$

which is one equation in one unknown t. As an exercise, the reader will be asked to find an exact expression for t.

As a first approximation, t is often calculated as a weighted average of the various times of payment, where the weights are the various amounts paid, i.e.

$$\bar{t} = \frac{s_1t_1 + s_2t_2 + \cdots + s_nt_n}{s_1 + s_2 + \cdots + s_n} = \frac{\sum_{k=1}^{n} s_kt_k}{\sum_{k=1}^{n} s_k}. \tag{2.9}$$

This approximation is denoted by $\bar{t}$ and is often called using the *method of equated time*.

It is possible to prove that the value of $\bar{t}$ is always greater than the true value of t, or, alternatively, that the present value using the method of equated time is smaller than the true present value.

Consider s_1 quantities each equal to v^{t_1}, s_2 quantities each equal to v^{t_2}, and so forth until there are s_n quantities each equal to v^{t_n}. The arithmetic mean of these quantities is

$$\frac{s_1v^{t_1} + s_2v^{t_2} + \cdots + s_nv^{t_n}}{s_1 + s_2 + \cdots + s_n}.$$

The geometric mean of these quantities is

$$v^{\frac{s_1t_1 + s_2t_2 + \cdots + s_nt_n}{s_1 + s_2 + \cdots + s_n}} = v^{\bar{t}}.$$

However, we know that the arithmetic mean of n positive numbers, not all of which are equal, is greater than the geometric mean, and thus we have

$$\frac{s_1v^{t_1} + s_2v^{t_2} + \cdots + s_nv^{t_n}}{s_1 + s_2 + \cdots + s_n} > v^{\bar{t}}$$

or

$$s_1 v^{t_1} + s_2 v^{t_2} + \cdots + s_n v^{t_n} > (s_1 + s_2 + \cdots + s_n) v^{\bar{t}}.$$

The left-hand side is the true present value which exceeds the present value given by the method of equated time on the right-hand side. Thus, the value of $\bar{t}$ from formula (2.9) is always greater than the true value of t from formula (2.8).

The method of equated time is useful in analyzing the average length of financial transactions. This application will be discussed further in Section 9.8.

Another interesting question often asked is how long it takes money to double at a given rate of interest. We can analyze this problem as follows:

$$(1 + i)^n = 2$$

or

$$n \log_e(1 + i) = \log_e 2$$

giving

$$n = \frac{\log_e 2}{\log_e(1 + i)}. \tag{2.10}$$

It is possible to derive an approximation to the exact result given in formula (2.10) as follows:

$$\begin{aligned} n &= \frac{\log_e 2}{\log_e(1 + i)} \\ &= \frac{.6931}{i} \cdot \frac{i}{\log_e(1 + i)}. \end{aligned}$$

The second factor evaluated for $i = 8\%$ is 1.0395. Thus we have

$$\begin{aligned} n &\doteq \frac{.6931}{i}(1.0395) \\ &= \frac{.72}{i}. \end{aligned} \tag{2.11}$$

This is frequently called the *rule of 72*, since it can be applied immediately by dividing 72 by the rate of interest expressed as a percentage (i.e. as $100i$).

The rule of 72 produces surprisingly accurate results over a wide range of interest rates. Illustrative values are provided in Table 2.1.

Table 2.1 Length of Time It Takes Money to Double

Rate of interest	*Rule of 72*	*Exact value*
4%	18	17.67
6	12	11.90
8	9	9.01
10	7.2	7.27
12	6	6.12
18	4	4.19

***Example 2.5** Find the length of time necessary for \$1000 to accumulate to \$1500 if invested at 6% per annum compounded semiannually: (1) by use of logarithms, and (2) by interpolating in the interest tables.*

Let n be the number of half-years. The equation of value is

$$1000(1.03)^n = 1500$$
$$(1.03)^n = 1.5.$$

1. Using logarithms

$$n \log_e 1.03 = \log_e 1.5$$
$$n = \frac{\log_e 1.5}{\log_e 1.03} = \frac{.405465}{.029559} = 13.717.$$

Thus, the number of years is 6.859.

2. From the interest tables, $(1.03)^{13} = 1.46853$ and $(1.03)^{14} = 1.51259$, so that $13 < n < 14$. Performing a linear interpolation

$$n = 13 + \frac{1.50000 - 1.46853}{1.51259 - 1.46853} = 13.714.$$

Thus, the number of years is 6.857. This method is equivalent to the assumption of simple interest during the final fraction of an interest conversion period. This point was discussed in more detail in Section 2.2.

***Example 2.6** Payments of \$100, \$200, and \$500 are due at the ends of years 2, 3, and 8, respectively. Assuming an effective rate of interest of 5% per annum, find the point in time at which a payment of \$800 would be equivalent: (1) by the method of equated time, and (2) by an exact method.*

1. By the method of equated time using formula (2.9)

$$\bar{t} = \frac{100 \cdot 2 + 200 \cdot 3 + 500 \cdot 8}{100 + 200 + 500} = 6 \text{ years.}$$

2. The exact equation of value is

$$800 v^t = 100 v^2 + 200 v^3 + 500 v^8$$

or

$$v^t = \frac{100(.90703) + 200(.86384) + 500(.67684)}{800} = .75236$$

which can be solved for t

$$t = -\frac{\log_e .75236}{\log_e 1.05} = -\frac{-.28454}{.04879} = 5.832 \text{ years.}$$

As expected, the true value of t is less than the value using the method of equated time.

2.7 UNKNOWN RATE OF INTEREST

Section 2.6 considered the case in which the length of the investment period is the unknown. In this section we consider the situation in which the rate of interest is the unknown. Problems involving the determination of an unknown rate of interest are widely encountered in practice, since it is often necessary to compute the rate of return involved in a particular transaction. Techniques to solve for an unknown rate of interest will be considered again in several later chapters for various important types of applications.

We consider four methods to use in determining an unknown rate of interest. The first of these is to solve the equation of value for i directly using a calculator with exponential and logarithmic functions. This method will work well if a single payment is involved and can occasionally be adapted to other situations as well.

The second method is to solve the equation of value for i by algebraic techniques. For example, an equation of value with integral exponents on all the terms can be written as an nth degree polynomial in i. If the roots of this polynomial can be determined algebraically, then i is immediately determined. This method is generally practical only for small values of n.

The third method is to use linear interpolation in the interest tables. This technique was illustrated in Section 2.6 for determining unknown time and can be adapted for problems involving an unknown rate of interest as well.

The fourth method is *successive approximation* or *iteration*. In using iteration to solve for an unknown rate of interest, a function involving i, denoted by $f(i)$, is determined using an equation of value, and iteration is used to find a value of i such that $f(i) = 0$. The number of successive approximations will be greatly reduced if a starting value close to the true root is used. Good starting values can often be obtained by linear interpolation in the interest tables, i.e. by the third method described above. Iteration methods are discussed in more detail in Appendix V and these methods will be utilized in later chapters

for some commonly-encountered applications. An ad hoc iteration technique will be illustrated in Example 2.10.

Example 2.7 ***At what interest rate convertible quarterly would $1000 accumulate to $1600 in six years?***

Let $j = i^{(4)}/4$ so that the equation of value becomes

$$1000(1 + j)^{24} = 1600$$

or

$$j = (1.6)^{1/24} - 1 .$$

This equation can be solved directly, which gives

$$j = .019776.$$

The answer is

$$i^{(4)} = 4j = .0791, \text{ or } 7.91\%.$$

Example 2.8 ***At what effective rate of interest will the present value of $2000 at the end of two years and $3000 at the end of four years be equal to $4000?***

An equation of value is

$$4000 = 2000v^2 + 3000v^4$$

which can be rewritten as

$$3v^4 + 2v^2 - 4 = 0 .$$

This equation can be solved as a quadratic in v^2, which gives

$$v^2 = \frac{-2 \pm \sqrt{4 + 4 \cdot 3 \cdot 4}}{2 \cdot 3} .$$

Since $v^2 > 0$, only the positive root is meaningful, and we have

$$9v^2 = \frac{-2 + \sqrt{52}}{6} = .868517$$

or

$$(1 + i)^2 = 1.151388$$

which gives

$$i = .0730, \text{ or } 7.30\% .$$

Example 2.9 ***At what interest rate convertible semiannually would an investment of $1000 immediately and $2000 3 years from now accumulate to $5000 10 years from now?***

Let $j = i^{(2)}/2$ so that the equation of value becomes

$$1000(1+j)^{20} + 2000(1+j)^{14} = 5000.$$

This problem cannot be solved analytically like the prior two examples, so we will use linear interpolation in the interest tables. Define

$$f(j) = 1000(1+j)^{20} + 2000(1+j)^{14} - 5000.$$

We seek to find j, such that $f(j) = 0$. By trial and error

$$f(.0300) = 1000(1.80611) + 2000(1.51259) - 5000 = -168.71$$
$$f(.0350) = 1000(1.98979) + 2000(1.61869) - 5000 = 227.17$$

and performing a linear interpolation

$$j = .0300 + .0050\,\frac{0 + 168.71}{227.17 + 168.71} = .0321$$

which gives

$$i^{(2)} = 2(.0321) = .0642, \text{ or } 6.42\%.$$

Example 2.10 Obtain the answer to Example 2.9 to a higher level of accuracy using iteration.

We will use a calculator with exponential and logarithmic functions and an ad hoc iteration method. From Example 2.9 we have

$$f(j) = 1000(1+j)^{20} + 2000(1+j)^{14} - 5000$$

and a first approximation $j = .0321$. Note that $f(j)$ is an increasing function of j for $j > 0$. We have

$$f(.0321) = -6.114$$
$$f(.0322) = 1.759.$$

We used a higher value than .0321 in order to obtain a sign change for $f(i)$.

Performing another linear interpolation,

$$j = .0321 + .0001\,\frac{0 + 6.114}{1.759 + 6.114} = .03218.$$

Cycling again with one more decimal place

$$f(.03218) = .18346$$
$$f(.03217) = -.60420.$$

In this case we used a lower value than .03218 in order to obtain a sign change for $f(i)$.

Performing another linear interpolation

$$j = .03217 + .00001\,\frac{0 + .60420}{.18346 + .60420} = .032178.$$

Cycling again with one more decimal place

$$f(.032178) = .025919$$
$$f(.032177) = -.052851.$$

Thus, $j = .032178$, which is accurate to six decimal places, since

$$| f(.032178) | < | f(.032177) |$$

The more accurate answer is

$$i^{(2)} = 2(.032178) = .06436, \text{ or } 6.436\%.$$

The above procedure can be repeated as many times as necessary. One additional decimal place of accuracy is achieved on each cycle.

Although methods with faster rates of convergence do exist (see Appendix V), the above technique is simple and straightforward, and has a more rapid rate of convergence than most ad hoc methods. Moreover, a similar approach can be developed for a wide variety of problems in interest involving the determination of unknown rates of interest.

2.8 PRACTICAL EXAMPLES

Virtually everyone is exposed to interest calculations regularly in their financial affairs. However, in practice interest calculations do not always follow the exact procedures outlined in this book. The purpose of this section is to familiarize the reader with some of the variations encountered in practice. The examples in this section are by no means exhaustive, but they are illustrative of some "real-world" applications of interest. Some terms for types of investments with which the reader may be unfamiliar are used in this section, e.g. "certificate of deposit" and "money market fund." Definitions of these terms are given in Section 8.8.

One type of advertisement frequently seen in the newspapers quotes two different rates on deposits. As this is being written, the author is looking at two such advertisements. In the first, a bank is quoting a one-year certificate of deposit as having "7.91% rate/8.15% yield." In the second, a savings and loan is quoting a money market fund as having "8.00% interest rate/8.30% annual yield." What is the meaning attached to these two numbers? In the above illustrations the former number is a nominal rate, while the latter is an annual effective rate. The reader should verify that $i^{(4)} = .0791$ is equivalent to $i = .0815$ and that $i^{(12)} = .0800$ is equivalent to $i = .0830$. In the former case the frequency of compounding was stated in the advertisement, but in the latter case it was not. The monthly frequency was ascertained by trial and error.

Section 2.3 deals with some of the variations in counting days. One advertisement the author has seen indicates that a savings bank credits 6% compounded daily which produces a yield of 6.27%. Using either a 360 or 365-day year produces an answer of 6.18%. How can an answer of 6.27% be obtained? After some trial and error, it was discovered that the savings bank was using a mix of 360-day and 365-day years as follows:

$$\left(1 + \frac{.06}{360}\right)^{365} - 1 = .0627.$$

When interest is not compounded daily, some interesting variations appear. Consider an investment account which credits interest monthly and on which deposits and withdrawals are occurring. One common variation credits interest on the average daily balance. Another common variation credits interest on the minimum daily balance. Yet a third variation credits interest on the beginning balance, reduced by any withdrawals, but not incremented by any deposits. In this latter case deposits do not start earning interest until the beginning of the following month.

Even the frequency of compounding can be tricky. Usually compounding occurs at some regular, predetermined interval. However, the author has encountered one money market fund which compounded interest whenever the interest rate being credited was changed. Thus, in a month with unchanged rates interest would be compounded only once, while in another more volatile month interest might be compounded as many as six or more times.

It is also important to distinguish between rates of interest and rates of discount. For example, the United States Treasury issues Treasury bills for periods of 13, 26, and 52 weeks. The rates on "T-bills," as they are often called, are computed as rates of discount. On the other hand, rates on longer-term Treasury securities are computed as rates of interest. Thus, rates on Treasury bills cannot be directly compared with rates on longer-term Treasury securities unless converted to equivalent rates. Treasury bills are discussed further in Section 7.2.

Rates of discount are also encountered in short-term commercial transactions. For example, if \$10,000 is borrowed on a discount basis at 12% for one month, then the borrower receives \$9900 immediately and repays \$10,000 at the end of the month. Such short-term transactions are often computed on a simple discount basis, as just illustrated. The net effect is equivalent to using a rate of discount convertible at the same frequency as the term of the loan. Thus, if a stated rate, such as the 12% used above, is used for loans of varying lengths, then the equivalent effective rates will differ depending upon the length of the loans.

Credit cards have an interesting way of charging interest. The interest generally is computed on the ending balance of the prior month. Thus, no interest is assessed on new charges during a month for that month, i.e. interest does not begin to be charged until the following month. In essence, this is an interest-free loan from the date of charge until the end of the month. Persons who always pay their credit card charges in full each month are, in essence,

borrowing money for short periods of time without paying any interest at all. On the other hand, the interest rate charged on those accounts which do carry outstanding balances from month-to-month tends to be very high.

Another feature that investors need to be careful to consider is a penalty for early withdrawal. For example, such penalties apply to many certificates of deposit. If a two-year certificate of deposit credits an annual rate of 9%, the purchaser is likely to have a rude awakening upon surrendering after one year and learning that less than 9% will be paid. The penalties for early withdrawal often involve either a reduction in the credited interest rate, or not crediting interest for a certain period of time, or some combination of the two. Other types of early withdrawal penalties are also encountered.

These illustrations should give the reader a flavor of some of the types of applications encountered. The fact that so many variations exist and that consistent terminology is not used from situation to situation can make seemingly simple financial analysis difficult. In real-world applications the reader is cautioned to look below the surface and ascertain exactly how calculations involving interest actually will be done. Otherwise, comparisons among various options for borrowing and lending may well be flawed.

EXERCISES

2.2 Obtaining numerical results

1. \$10,000 is invested for four months at 12.6%, where interest is computed using a quadratic to approximate an exact calculation. Find the accumulated value.

2. Find the present value of \$5000 to be paid at the end of 25 months at a rate of discount of 8% convertible quarterly:
   a) Assuming compound discount throughout.
   b) Assuming simple discount during the final fractional period.

3. Find an expression for the fraction of a period at which the excess of present values computed at simple discount over compound discount is a maximum.

4. Method A assumes simple interest over final fractional periods, while Method B assumes simple discount over final fractional periods. The annual effective rate of interest is 20%. Find the ratio of the present value of a payment to be made in 1.5 years computed under Method A to that computed under Method B.

2.3 Determining time periods

5. If an investment was made on the day the United States entered World War II, i.e. December 7, 1941, and was terminated at the end of the war on August 8, 1945, for how many days was the money invested:

a) On the actual/actual basis?
b) On the 30/360 basis?

6. A sum of \$10,000 is invested for the months of July and August at 6% simple interest. Find the amount of interest earned:
   a) Assuming exact simple interest.
   b) Assuming ordinary simple interest.
   c) Assuming the Banker's Rule.

7. *a*) Show that the Banker's Rule is always more favorable to the lender than is exact simple interest.
   b) Show that the Banker's Rule is usually more favorable to the lender than is ordinary simple interest.
   c) Find a counterexample in (*b*) for which the opposite relationship holds.

2.5 Equations of value

8. In return for payments of \$2000 at the end of four years and \$5000 at the end of ten years, an investor agrees to pay \$3000 immediately and to make an additional payment at the end of three years. Find the amount of the additional payment if $i^{(4)} = .06$.

9. At a certain interest rate the present value of the following two payment patterns are equal:
   (*i*) \$200 at the end of 5 years plus \$500 at the end of 10 years.
   (*ii*) \$400.94 at the end of 5 years.
   At the same interest rate \$100 invested now plus \$120 invested at the end of 5 years will accumulate to P at the end of 10 years. Calculate P.

10. An investor makes three deposits into a fund, at the end of 1, 3, and 5 years. The amount of the deposit at time t is $100(1.025)^t$. Find the size of the fund at the end of 7 years, if the nominal rate of discount convertible quarterly is 4/41.

11. Whereas the choice of a comparison date has no effect on the answer obtained with compound interest, the same cannot be said of simple interest. Find the amount to be paid at the end of 10 years which is equivalent to two payments of \$100 each, the first to be paid immediately and the second to be paid at the end of 5 years. Assume 5% simple interest is earned from the date each payment is made and use a comparison date of:
   a) The end of 10 years.
   b) The end of 15 years.

2.6 Unknown time

12. Solve formula (2.8) exactly.

13. Find how long \$1000 should be left to accumulate at 6% effective in order that it will amount to twice the accumulated value of another \$1000 deposited at the same time at 4% effective.

14. The present value of two payments of \$100 each to be made at the end of n years and $2n$ years is \$100. If $i = .08$, find n.

15. A payment of n is made at the end of n years, $2n$ at the end of $2n$ years, . . . , n^2 at the end of n^2 years. Find the value of t by the method of equated time.

16. You are asked to develop a *rule of n* to approximate how long it takes money to triple. Find n.

17. A loan is negotiated with the lender agreeing to accept \$1000 after 10 years, \$2000 after 20 years, and \$3000 after 30 years in full repayment. The borrower wishes to liquidate the loan with a single \$6000 payment. Let T_1 represent the time of the \$6000 payment calculated by an equation of value. Let T_2 represent the time determined by the method of equated time. If $i = .01$, find $T_1 - T_2$ to the nearest .10. Use linear interpolation in the tables.

18. Fund A accumulates at a rate of 12% convertible monthly. Fund B accumulates with a force of interest $\delta_t = t/6$. At time $t = 0$ equal deposits are made in each fund. Find the next time that the two funds are equal.

2.7 Unknown rate of interest

19. Find the nominal rate of interest convertible seminannually at which the accumulated value of \$1000 at the end of 15 years is \$3000.

20. Find an expression for the exact effective rate of interest at which payments of \$300 at the present, \$200 at the end of one year, and \$100 at the end of two years will accumulate to \$700 at the end of two years.

21. It is known that an investment of \$1000 will accumulate to \$1825 at the end of 10 years. If it is assumed that the investment earns simple interest at rate i during the 1st year, $2i$ during the 2nd year, . . . , $10i$ during the 10th year, find i.

22. It is known that an amount of money will double itself in 10 years at a varying force of interest $\delta_t = kt$. Find an expression for k.

23. The sum of the accumulated value of 1 at the end of three years at a certain effective rate of interest, and the present value of 1 to be paid at the end of three years at an effective rate of discount numerically equal to i is 2.0096. Find the rate.

24. Rework Examples 2.9 and 2.10 using the following formula to generate successive values:

$$j_{s+1} = j_s - \frac{f(j_s)}{f'(j_s)}.$$

Show that the answer obtained in Example 2.10 is obtained as j_1 after only one iteration using the answer obtained in Example 2.9 as the starting value j_0.

2.8 Practical examples

25. A bill for \$100 is purchased for \$96 three months before it is due. Find:
    a) The nominal rate of discount convertible quarterly earned by the purchaser.
    b) The annual effective rate of interest earned by the purchaser.

26. A two-year certificate of deposit pays an annual effective rate of 9%. The purchaser is offered two options for prepayment penalties in the event of early withdrawal:
    A – a reduction in the rate of interest to 7%.
    B – loss of three months interest.
    In order to assist the purchaser in deciding which option to select, compute the ratio of the proceeds under Option A to those under Option B if the certificate of deposit is surrendered:
    a) At the end of 6 months.
    b) At the end of 18 months.

27. A savings and loan association pays 7% effective on deposits at the end of each year. At the end of every three years a 2% bonus is paid on the balance at that time. Find the effective rate of interest earned by an investor if the money is left on deposit:
    a) Two years.
    b) Three years.
    c) Four years.

28. A bank offers the following certificates of deposit:

Term in years	Nominal annual interest rate (convertible semiannually)
1	5%
2	6%
3	7%
4	8%

The bank does not permit early withdrawal. The certificates mature at the end of the term. During the next six years the bank will continue to offer these certificates of deposit. An investor deposits \$1000 in the bank. Calculate the maximum amount that can be withdrawn at the end of six years.

Miscellaneous problems

29. A manufacturer sells a product to a retailer who has the option of paying 30% below the retail price immediately, or 25% below the retail price in six months. Find the annual effective rate of interest at which the retailer would be indifferent between the two options.

30. If an investment will be doubled in 8 years at a force of interest δ, in how many years will an investment be tripled at a nominal rate of interest numerically equal to δ and convertible once every three years?

31. Fund A accumulates at 6% effective and Fund B accumulates at 8% effective. At the end of 20 years the total of the two funds is \$2000. At the end of 10 years the amount in Fund A is half that in Fund B. What is the total of the two funds at the end of 5 years?

32. An investor deposits \$10,000 in a bank. During the first year, the bank credits an annual effective rate of interest i. During the second year, the bank credits an annual effective rate of interest $i - .05$. At the end of two years the account balance is \$12,093.75. What would the account balance have been at the end of three years, if the annual effective rate of interest were $i + .09$ for each of the three years?

33. A signs a one-year note for \$1000 and receives \$920 from the bank. At the end of six months A makes a payment of \$288. Assuming simple discount, to what amount does this reduce the face amount of the note?

3

Basic annuities

3.1 INTRODUCTION

An *annuity* may be defined as a series of payments made at equal intervals of time. Annuities are common in our economic life. House rents, mortgage payments, installment payments on automobiles, and interest payments on money invested are all examples of annuities. Originally the meaning of the word "annuity" was restricted to annual payments, but it has been extended to include payments made at other regular intervals as well.

Consider an annuity such that payments are certain to be made for a fixed period of time. An annuity with these properties is called an *annuity-certain*. The fixed period of time for which payments are made is called the *term* of the annuity. For example, mortgage payments on a home or business constitute an annuity-certain.

Not all annuities are annuities-certain. An annuity under which the payments are not certain to be made is called a *contingent annuity*. A common type of contingent annuity is one in which payments are made only if a person is alive. Such an annuity is called a *life annuity*. For example, monthly retirement benefits from a pension plan, which continue for the life of a retiree, constitute a life annuity.

In this book, we will largely restrict our attention to annuities-certain. However, the risk of default or non-payment is addressed in Section 9.5. It will often be convenient to drop the word "certain" and use the term "annuity" to refer to an annuity-certain.

The interval between annuity payments is called the *payment period*. In this chapter, we consider annuities for which the payment period and the interest conversion period are equal and coincide, and for which the payments are of

level amount. Since the payment period and the interest conversion period are equal and coincide, we will just use the term "period" for both. In Chapter 4, annuities for which payments are made more or less frequently than interest is converted and annuities with varying payments are examined.

3.2 ANNUITY-IMMEDIATE

Consider an annuity under which payments of 1 are made at the end of each period for n periods. Such an annuity is called an *annuity-immediate*. Figure 3.1 is a time diagram for such an annuity. Arrow 1 appears one period before the first payment is made. We assume that the rate of interest is i per period. The present value of the annuity at this point in time is denoted by $a_{\overline{n}|}$. Arrow 2 appears n periods after arrow 1, just after the last payment is made. The accumulated value of the annuity at this point in time is denoted by $s_{\overline{n}|}$.

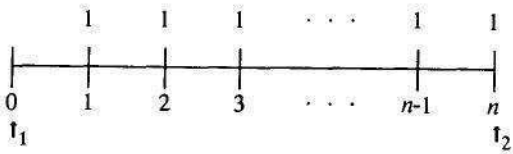

Figure 3.1 Time diagram for an annuity-immediate

We can derive an expression for $a_{\overline{n}|}$ as an equation of value at the beginning of the first period. The present value of a payment of 1 made at the end of the first period is v. The present value of a payment of 1 made at the end of the second period is v^2. This process is continued until the present value of a payment of 1 made at the end of the nth period is v^n. The total present value $a_{\overline{n}|}$ must equal the sum of the present values of each payment, i.e.

$$a_{\overline{n}|} = v + v^2 + \cdots + v^{n-1} + v^n. \tag{3.1}$$

This formula could be used to evaluate $a_{\overline{n}|}$, but it would become inefficient for large n. It is possible to derive a more compact expression by recognizing that formula (3.1) is a geometric progression

$$\begin{aligned} a_{\overline{n}|} &= v + v^2 + \cdots + v^{n-1} + v^n \\ &= v\frac{1 - v^n}{1 - v} \\ &= v\frac{1 - v^n}{iv} \\ &= \frac{1 - v^n}{i}. \end{aligned} \tag{3.2}$$

An expression for $s_{\overline{n}|}$ can be derived in an analogous manner as an equation of value at the end of the nth period. The accumulated value of a payment of 1 made at the end of the first period is $(1+i)^{n-1}$. The accumulated value of a payment of 1 made at the end of the second period is $(1+i)^{n-2}$. This process is continued until the accumulated value of a payment of 1 made at the end of the nth period is just 1. The total accumulated value $s_{\overline{n}|}$ must equal the sum of the accumulated values of each payment, i.e.

$$s_{\overline{n}|} = 1 + (1+i) + \cdots + (1+i)^{n-2} + (1+i)^{n-1}. \tag{3.3}$$

Again, a more compact expression can be derived by summing the geometric progression

$$\begin{aligned} s_{\overline{n}|} &= 1 + (1+i) + \cdots + (1+i)^{n-2} + (1+i)^{n-1} \\ &= \frac{(1+i)^n - 1}{(1+i) - 1} \\ &= \frac{(1+i)^n - 1}{i}. \end{aligned} \tag{3.4}$$

Values of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ at several rates of interest and for values of n from 1 to 50 appear in the interest tables in Appendix I. On occasion, the interest rate is written to the lower right of the symbol, e.g. $a_{\overline{10}|.07}$ and $s_{\overline{25}|.08}$. Since this tends to clutter the symbols, we will do this only if there could be any ambiguity concerning the interest rate to be used in evaluating the function.

It is possible to give a verbal interpretation to formula (3.2) written as

$$1 = ia_{\overline{n}|} + v^n.$$

Consider the investment of 1 for n periods. Each period the investment of 1 will yield interest of i paid at the end of the period. The present value of these interest payments is $ia_{\overline{n}|}$. At the end of n periods the original investment of 1, whose present value is v^n, is returned. Thus, both sides of the equation represent the present value of an investment of 1 at the date of investment. A similar verbal interpretation for formula (3.4) is left as an exercise.

There is a simple relationship between $a_{\overline{n}|}$ and $s_{\overline{n}|}$

$$s_{\overline{n}|} = a_{\overline{n}|}(1+i)^n. \tag{3.5}$$

This relationship is obvious from a comparison of either formulas (3.1) and (3.3) or from formulas (3.2) and (3.4). It is also obvious from the time diagram, since $s_{\overline{n}|}$ is the value of the same payments as $a_{\overline{n}|}$, only the value is taken n periods later.

Another relationship between $a_{\overline{n}|}$ and $s_{\overline{n}|}$ is

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i. \tag{3.6}$$

This relationship can be derived as follows:

$$\begin{aligned}\frac{1}{s_{\overline{n}|}} + i &= \frac{i}{(1+i)^n - 1} + i \\ &= \frac{i + i(1+i)^n - i}{(1+i)^n - 1} \\ &= \frac{i}{1 - v^n} \\ &= \frac{1}{a_{\overline{n}|}}.\end{aligned}$$

Values of $1/s_{\overline{n}|}$ appear in the interest tables in Appendix I. Formula (3.6) will be quite significant in another context in Chapter 6. A verbal interpretation of the formula will be given at that time.

***Example 3.1** Find the present value of an annuity which pays \$500 at the end of each half-year for 20 years if the rate of interest is 9% convertible semiannually.*

The answer is

$$500\,a_{\overline{40}|.045} = 500(18.4016) = \$9200.80.$$

***Example 3.2** If a person invests \$1000 at 8% per annum convertible quarterly, how much can be withdrawn at the end of every quarter to use up the fund exactly at the end of 10 years?*

Let R be the amount of each withdrawal. The equation of value at the date of investment is

$$R\,a_{\overline{40}|.02} = 1000.$$

Thus, we have

$$\begin{aligned}R &= \frac{1000}{a_{\overline{40}|.02}} \\ &= \frac{1000}{27.3555} \\ &= \$36.56\end{aligned}$$

***Example 3.3** Compare the total amount of interest that would be paid on a \$1000 loan over a 10-year period, if the effective rate of interest is 9% per annum, under the following three repayment methods:*

(1) The entire loan plus accumulated interest is paid in one lump-sum at the end of 10 years.

(2) Interest is paid each year as accrued and the principal is repaid at the end of 10 years.

(3) The loan is repaid by level payments over the 10-year period.

1. The accumulated value of the loan at the end of 10 years is

$$1000(1.09)^{10} = \$2367.36$$

Thus, the total amount of interest paid is equal to

$$\$2367.36 - 1000.00 = \$1367.36.$$

2. Each year the loan earns interest of $1000(.09) = \$90$, so that the total amount of interest paid is equal to

$$10 \cdot 90 = \$900.00.$$

3. Let the level payment be R. An equation of value for R at the inception of the loan is

$$R\,a_{\overline{10}|} = 1000$$

which gives

$$R = \frac{1000}{a_{\overline{10}|}} = \frac{1000}{6.417658} = \$155.82.$$

Thus, the total amount of interest paid is equal to

$$10(155.82) - 1000 = \$558.20.$$

The reader should justify by general reasoning the relative answers under the three repayment methods. The later the payments are made on a loan, the higher the total amount of interest will be. Conversely, the sooner the payments are made, the smaller the amount of interest. Although the total amount of payments under the three methods are different, the present value of payments is equal to \$1000, the amount of the loan, for all three.

3.3 ANNUITY-DUE

In Section 3.2 the annuity-immediate was defined as an annuity in which payments are made at the end of the period. In this section, we will consider the *annuity-due* in which payments are made at the beginning of the period instead. The use of the terms "annuity-immediate" and "annuity-due" is unfortunate, since these terms are not descriptive of the properties of these annuities.

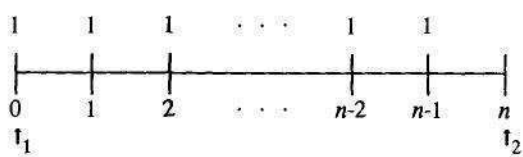

Figure 3.2 Time diagram for an annuity-due

Figure 3.2 is a time diagram for an n-period annuity-due. Arrow 1 appears at the time the first payment is made. The present value of the annuity at this point in time is denoted by $\ddot{a}_{\overline{n}|}$. Arrow 2 appears n periods after arrow 1, one period after the last payment is made. The accumulated value of the annuity at this point in time is denoted by $\ddot{s}_{\overline{n}|}$.

We can write an expression for $\ddot{a}_{\overline{n}|}$ analogous to formula (3.1)

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + \cdots + v^{n-1}. \tag{3.7}$$

Again summing the geometric progression

$$\begin{aligned} \ddot{a}_{\overline{n}|} &= \frac{1 - v^n}{1 - v} \\ &= \frac{1 - v^n}{iv} \\ &= \frac{1 - v^n}{d} \end{aligned} \tag{3.8}$$

which is analogous to formula (3.2).

Similarly for $\ddot{s}_{\overline{n}|}$, we have the following formulas analogous to formulas (3.3) and (3.4)

$$\ddot{s}_{\overline{n}|} = (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-1} + (1 + i)^n \tag{3.9}$$

$$\begin{aligned} &= (1 + i)\frac{(1 + i)^n - 1}{(1 + i) - 1} \\ &= \frac{(1 + i)^n - 1}{iv} \\ &= \frac{(1 + i)^n - 1}{d}. \end{aligned} \tag{3.10}$$

It is instructive to compare formulas (3.2) and (3.8). The numerators are identical; however, the denominator of (3.2) is i and the denominator of (3.8) is d. Under the annuity-immediate, payments are made at the end of the period and i is a measure of interest payable at the end of the period. Under the annuity-due, payments are made at the beginning of the period and d is a measure of interest payable at the beginning of the period. A comparison of formulas (3.4) and (3.10) gives similar results.

The above property, relating the time annuity payments are made to the measure of interest in the denominator, is helpful in remembering annuity formulas. Moreover, this property can be generalized to the more complex annuities which are discussed in Chapter 4.

It is immediately obvious that

$$\ddot{s}_{\overline{n}|} = \ddot{a}_{\overline{n}|}(1+i)^n \tag{3.11}$$

a formula analogous to formula (3.5). It can also be shown that

$$\frac{1}{\ddot{a}_{\overline{n}|}} = \frac{1}{\ddot{s}_{\overline{n}|}} + d \tag{3.12}$$

a formula analogous to formula (3.6). The derivation of formula (3.12) is left as an exercise.

It is possible to relate the annuity-immediate and the annuity-due. One type of relationship is

$$\ddot{a}_{\overline{n}|} = a_{\overline{n}|}(1+i) \tag{3.13}$$

and

$$\ddot{s}_{\overline{n}|} = s_{\overline{n}|}(1+i)\,. \tag{3.14}$$

Formula (3.13) can immediately be derived by comparing formulas (3.1) and (3.7), or formulas (3.2) and (3.8). Since each payment under $\ddot{a}_{\overline{n}|}$ is made one period earlier than each payment under $a_{\overline{n}|}$, the total present value must be larger by one period's interest. Formula (3.14) can be derived similarly.

There is another type of relationship between the annuity-immediate and the annuity-due

$$\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}. \tag{3.15}$$

This formula can be derived from Figure 3.2. The n payments made under $\ddot{a}_{\overline{n}|}$ can be split into the first payment and the remaining $n-1$ payments. The present value of the first payment is 1, and the present value of the remaining $n-1$ payments is $a_{\overline{n-1}|}$. The sum must give the total present value $\ddot{a}_{\overline{n}|}$.

Similarly, we can obtain

$$\ddot{s}_{\overline{n}|} = s_{\overline{n+1}|} - 1\,. \tag{3.16}$$

This formula can also be derived from Figure 3.2. Temporarily, assume that an imaginary payment of 1 is made at the end of the nth period. Then, the total accumulated value of the $n+1$ payment is $s_{\overline{n+1}|}$. However, we must remove the accumulated value of the imaginary payment, which is just 1. The difference gives the accumulated value $\ddot{s}_{\overline{n}|}$.

Most compound interest tables, including those in Appendix I, do not include values for annuities-due. Thus, formulas such as (3.13) or (3.15) and (3.14) or (3.16) must be used in finding numerical values for annuities-due.

Considerable confusion has often been created by treating the annuity-immediate and the annuity-due as if they were greatly different. Actually they refer to exactly the same series of payments evaluated at different points in time. Figure 3.3 clarifies this point.

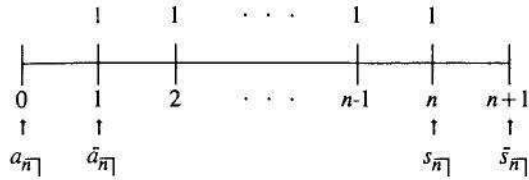

Figure 3.3 Time diagram comparing an annuity-immediate with an annuity-due

Example 3.4 *An investor wishes to accumulate $1000 in a fund at the end of 12 years. To accomplish this the investor plans to make deposits at the end of each year, the final payment to be made one year prior to the end of the investment period. How large should each deposit be if the fund earns 7% effective?*

Since we are interested in the accumulated value one year after the last payment, the equation of value is

$$R\ddot{s}_{\overline{11}|} = 1000$$

where R is the annual deposit. Solving for R we have

$$\begin{aligned} R &= \frac{1000}{\ddot{s}_{\overline{11}|}} \\ &= \frac{1000}{1.07 s_{\overline{11}|}} \\ &= \frac{1000}{(1.07)(15.7836)} = \$59.21. \end{aligned}$$

3.4 ANNUITY VALUES ON ANY DATE

Thus far we have considered evaluating annuities only at the beginning of the term (either one period before, or on the date of, the first payment), or at the end of the term (either on the date of, or one period after, the last payment). However, it is often necessary to evaluate annuities on other dates. We will discuss the following three cases: (1) present values more than one period before the first payment date, (2) accumulated values more than one period after the last payment date, and (3) current values between the first and last payment dates. We will assume that the evaluation date is an integral number of periods from each payment date.

The value of an annuity on any date could be found by accumulating or discounting each separate payment and summing the results. However, this method would become inefficient if a large number of payments are involved.

We will see that it is possible to develop values for all three cases in terms of annuity symbols already defined.

The above three cases can best be illustrated by example. Consider an annuity under which seven payments of 1 are made at the end of the 3rd through the 9th periods inclusive. Figure 3.4 is a time diagram for this annuity. The values at the end of the 2nd, 3rd, 9th, and 10th periods are given directly, either by annuities-immediate or by annuities-due, as labeled on the time diagram.

The present value at the beginning of the 1st period is an example of case 1, the accumulated value at the end of the 12th period is an example of case 2, and the current value at the end of the 6th period is an example of case 3. These three cases are denoted by arrows 1, 2, and 3, respectively, on the time diagram.

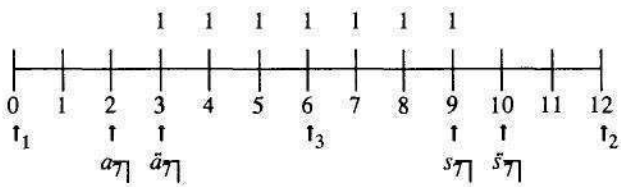

Figure 3.4 Time diagram for illustration in Section 3.4

Present values more than one period before the first payment date

In this case, the present value of the annuity at the beginning of the 1st period is seen to be the present value at the end of the 2nd period discounted for two periods, i.e.

$$v^2 a_{\overline{7}|}.$$

It is possible to develop an alternate expression for this present value strictly in terms of annuity values. Temporarily assume that imaginary payments of 1 are made at the end of the 1st and 2nd periods. Then the present value of all nine payments at time $t = 0$ is $a_{\overline{9}|}$. However, we must remove the present value of the imaginary payments, which is $a_{\overline{2}|}$. Thus, an alternate expression for the present value is

$$a_{\overline{9}|} - a_{\overline{2}|}.$$

This expression is convenient for calculation, if interest tables are being used.

This type of annuity is often called a *deferred annuity*, since payments commence only after a deferred period. In general, the present value of an annuity-immediate deferred for m periods with a term of n periods after the deferred period is

$$v^m a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}. \tag{3.17}$$

It should be noted that since payments are made at the ends of periods, the first payment under such a deferred annuity is $m + 1$ periods after the evaluation date, not m periods.

We have used annuities-immediate instead of annuities-due, since numerical values are directly available from the interest tables for annuities-immediate. However, it is possible to work with a deferred annuity-due. The reader should verify that the answer to this case, expressed as annuity-due, is

$$v^3 \ddot{a}_{\overline{7}|} = \ddot{a}_{\overline{10}|} - \ddot{a}_{\overline{3}|}.$$

Accumulated values more than one period after the last payment date

In this case, the accumulated value of the annuity at the end of the 12th period is seen to be the accumulated value at the end of the 9th period, accumulated for three periods, i.e.

$$s_{\overline{7}|}(1 + i)^3.$$

Here it is also possible to develop an alternate expression strictly in terms of annuity values. Temporarily, assume that imaginary payments of 1 are made at the end of the 10th, 11th, and 12th periods. Then the accumulated value of all 10 payments is $s_{\overline{10}|}$. However, we must remove the accumulated value of the imaginary payments, which is $s_{\overline{3}|}$. Thus, an alternate expression for the accumulated value is

$$s_{\overline{10}|} - s_{\overline{3}|}.$$

In general, the accumulated value of an n-period annuity, m periods after the last payment date, is

$$s_{\overline{n}|}(1 + i)^m = s_{\overline{m+n}|} - s_{\overline{m}|}. \tag{3.18}$$

It is also possible to work with annuities-due instead of annuities-immediate. The reader should verify that the answer in this case expressed as an annuity-due is

$$\ddot{s}_{\overline{7}|}(1 + i)^2 = \ddot{s}_{\overline{9}|} - \ddot{s}_{\overline{2}|}.$$

Current values between the first and last payment dates

In this case, the current value of the annuity at the end of the 6th period is seen to be the present value at the end of the 2nd period accumulated for four periods or the accumulated value at the end of the 9th period discounted for three periods, i.e.

$$a_{\overline{7}|}\,(1+i)^4 = v^3 s_{\overline{7}|}.$$

Here it is again possible to develop an alternate expression strictly in terms of annuity values. Separate the seven payments into the first four payments and the last three payments. The accumulated value of the first four payments is $s_{\overline{4}|}$ and the present value of the last three payments is $a_{\overline{3}|}$. Thus, an alternate expression for the current value is

$$s_{\overline{4}|} + a_{\overline{3}|}.$$

In general, the current value of an n-period annuity immediately after the mth payment has been made ($m < n$) is

$$a_{\overline{n}|}\,(1+i)^m = v^{n-m} s_{\overline{n}|} = s_{\overline{m}|} + a_{\overline{n-m}|}. \tag{3.19}$$

It is also possible to work with annuities-due instead of annuities-immediate. The reader should verify that the answer in this case expressed as an annuity-due is

$$\ddot{a}_{\overline{7}|}\,(1+i)^3 = v^4 \ddot{s}_{\overline{7}|} = \ddot{s}_{\overline{3}|} + \ddot{a}_{\overline{4}|}.$$

Summary

In general, it is possible to express the value of an annuity on any date which is an integral number of periods from each payment date as the sum or difference of annuities-immediate. Other equivalent expressions do exist, and the reader should practice translating one form of an answer into alternate forms.

The reader should not try to work problems by memorizing formulas (3.17), (3.18), and (3.19). Any problem of this type can be best handled from first principles, as illustrated in this section.

The reader should also be careful to observe that the labels on the time diagram are merely an aid in visualizing the payments involved, but they do not affect the answer. For example, we could label the time diagram in Figure 3.4 from 8 to 20 instead of 1 to 12 and the answers to the various examples would not change. The elements on the time diagram that do affect the answer are the

number of payments and the location of the evaluation date or comparison date in relation to the payment dates.

If it is necessary to find the value of an annuity on a date which is not an integral number of periods from each payment date, the value should be found on a date which is an integral number of periods from each payment date and then the value on this date can be accumulated or discounted for the fractional period to the actual evaluation date. This situation will be illustrated in the exercises.

3.5 PERPETUITIES

A *perpetuity* is an annuity whose payments continue forever, i.e. the term of the annuity is not finite. Although it seems unrealistic to have an annuity with payments continuing forever, examples do exist in practice. The dividends on preferred stock with no redemption provision and the British consols, which are nonredeemable obligations of the British government, are examples of perpetuities. Preferred stock will be defined and discussed further in Chapter 7.

The present value of a perpetuity-immediate is denoted by $a_{\overline{\infty}|}$, and we have

$$\begin{aligned} a_{\overline{\infty}|} &= v + v^2 + v^3 + \cdots \\ &= \frac{v}{1-v} \\ &= \frac{v}{iv} \\ &= \frac{1}{i} \end{aligned} \tag{3.20}$$

provided $v < 1$, which will be the case if $i > 0$.

Alternatively, we have

$$a_{\overline{\infty}|} = \lim_{n\to\infty} a_{\overline{n}|} = \lim_{n\to\infty} \frac{1-v^n}{i} = \frac{1}{i}$$

since

$$\lim_{n\to\infty} v^n = 0 .$$

Formula (3.20) can be interpreted verbally. If principal of $1/i$ is invested at rate i, then interest of $i \cdot 1/i = 1$ can be paid at the end of every period forever, leaving the original principal intact.

By an analogous argument, for a perpetuity-due, we have

$$\ddot{a}_{\overline{\infty}|} = \frac{1}{d} . \tag{3.21}$$

It should be noted that accumulated values for perpetuities do not exist, since payments continue forever.

It is instructive to use the concept of perpetuities to give a verbal interpretation to formula (3.2)

$$a_{\overline{n}|} = \frac{1 - v^n}{i}. \tag{3.2}$$

Consider two perpetuities. The first pays 1 at the end of each period and has present value $1/i$. The second is deferred n periods and, thus, has present value v^n/i. The difference is the present value of payments of 1 at the end of each period during the deferred period, which is $a_{\overline{n}|}$.

Some readers feel that perpetuities are of limited practical significance. However, a broader range of transactions than payments continuing forever can be looked upon as equivalent to perpetuities. For example, consider the investment of \$1000 at a 10% annual effective rate of interest, payable in installments each year, for any particular period of time, at the end of which the principal of \$1000 is returned. Such a transaction is equivalent to determining the present value of a perpetuity paying \$100 at the end of each year at an effective rate of interest of 10%.

***Example 3.5** A leaves an estate of \$100,000. Interest on the estate is paid to beneficiary B for the first 10 years, to beneficiary C for the second 10 years, and to charity D thereafter. Find the relative shares of B, C, and D in the estate, if it assumed the estate will earn a 7% annual effective rate of interest.*

The value of B's share is

$$7000\, a_{\overline{10}|} = 7000(7.0236) = \$49{,}165$$

to the nearest dollar.

The value of C's share is

$$7000(a_{\overline{20}|} - a_{\overline{10}|}) = 7000(10.5940 - 7.0236) = \$24{,}993$$

to the nearest dollar.

The value of D's share is

$$7000(a_{\overline{\infty}|} - a_{\overline{20}|}) = 7000\left[\frac{1}{.07} - 10.5940\right] = \$25{,}842$$

to the nearest dollar.

Note that the sum of the shares of B, C, and D is equal to \$100,000 as expected. Also note that the present value of the estate at the end of 20 years is $100{,}000(1.07)^{-20} = \$25{,}842$, to the nearest dollar, which is equal to D's share.

This confirms the fact that charity D continuing to receive the interest into perpetuity or receiving the estate value in a lump-sum at the end of 20 years are equivalent in value.

3.6 NONSTANDARD TERMS AND INTEREST RATES

Thus far we have assumed that n is a positive integer and that $i > 0$ in any of our annuity symbols. This section considers the implications if these conditions are not satisfied. Most of the results in this section are of greater mathematical than practical interest. However, the results in this section have appeared in the literature and thus need to be addressed. Also, the analysis is of instructive value.

Consider first what the symbol $a_{\overline{n+k}|}$, where n is a positive integer and $0 < k < 1$, might represent. Formula (3.1) cannot be applied, since it requires that n be a positive integer. However, it is possible to derive a result which is consistent with formula (3.2) as follows:

$$\begin{aligned} a_{\overline{n+k}|} &= \frac{1 - v^{n+k}}{i} \\ &= \frac{1 - v^n + v^n - v^{n+k}}{i} \\ &= a_{\overline{n}|} + v^{n+k}\left[\frac{(1+i)^k - 1}{i}\right]. \end{aligned} \tag{3.22}$$

Thus, an interpretation for the symbol $a_{\overline{n+k}|}$ consistent with the formula (3.2) is that it is the present value of an n-period annuity-immediate of 1 per period, plus a final payment at time $n + k$ of $\dfrac{(1+i)^k - 1}{i}$.

The amount of this final irregular payment seems rather unusual. A payment that might be more "comfortable" to some readers is k, i.e. the payment would be proportional to the fractional time involved. Fortunately, k is reasonably close to $\dfrac{(1+i)^k - 1}{i}$. As an exercise, the reader will be asked to find the error involved in this approximation.

However, it is possible to give other reasonable interpretations to the symbol $a_{\overline{n+k}|}$. For example, how about letting the final payment be k, but having it paid at time $n + 1$? Other possibilities also suggest themselves. Thus, $a_{\overline{n+k}|}$ is not really a well-defined symbol without further description of the payments actually involved. Formula (3.22) is one possible interpretation, but other interpretations are also reasonable.

Unfortunately, the need to interpret annuity symbols for non-integral terms does arise in practice. For example, many courts use an annuity-certain for a

person's life expectancy in the measurement of economic damages in personal injury and wrongful death lawsuits. If the injured party in such a lawsuit has a life expectancy of 15.7 years, then what is the correct value of the annuity-certain? This question does not have a clearcut answer and further interpretation is required.

It is also possible to develop identities for annuities in which n is negative. However, such annuities are only of mathematical concern and are devoid of any practical significance. Such identities are developed in the exercises for sake of completeness.

Consider next what happens if $i \leq 0$. The case in which $i = 0$ is important in practice. If $i = 0$, then the present value or accumulated value of any annuity is merely the sum of the payments. Thus, in particular, we have

$$a_{\overline{n}|} = s_{\overline{n}|} = n \text{ if } i = 0. \tag{3.23}$$

If $i < 0$, then some interesting results occur. Present values become accumulated values, and vice versa, which has intuitive appeal. These results are developed in the exercises. Again, however, these results are more of theoretical than practical significance.

3.7 UNKNOWN TIME

Thus far, in any problems involving annuities, we have assumed that n and i are both known. In Section 3.7 we will consider the case in which n is unknown, and in Section 3.8 we will consider the case in which i is unknown.

In general, problems involving unknown time will not produce exact integral answers for n. These problems could be handled along the lines of Section 3.6 in which a smaller payment is made during the period following the last regular payment.

However, in practice, this is seldom done because of the inconvenience and confusion of making a payment at a date which is not an integral number of periods from the dates all other payments are made. For example, making all regular payments on July 1 of each year for a period of years followed by a smaller payment on November 27 is not convenient for either party to the transaction.

What is usually done in practice is either to make a smaller additional payment at the same time as the last regular payment, in effect making a payment larger than the regular payment, called a ***balloon payment***, or to make a smaller payment one period after the last regular payment, called a ***drop payment***. Naturally, the smaller payments in these two situations are not equal,

nor would either be equal to the smaller payment made at an intermediate point as in Section 3.6. However, all these payments would be equivalent in value.

Problems involving unknown time can best be illustrated by example.

***Example 3.6** An investment of \$1000 is to be used to make payments of \$100 at the end of every year for as long as possible. If the fund earns an annual effective rate of interest of 5%, find how many regular payments can be made and find the amount of the smaller payment: (1) to be paid on the date of the last regular payment, (2) to be paid one year after the last regular payment, and (3) to be paid during the year following the last regular payment, as described in Section 3.6.*

The equation of value is

$$100\, a_{\overline{n}|} = 1000$$

or

$$a_{\overline{n}|} = 10.$$

By inspection of the interest tables, we have $14 < n < 15$. Thus, 14 regular payments can be made plus a smaller final payment. Figure 3.5 is a time diagram for this example.

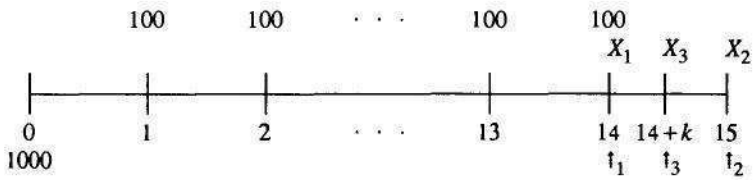

Figure 3.5 Time diagram for Example 3.6

In this figure X_1, X_2, and X_3 are the smaller final payments for the above three cases; arrows 1, 2, and 3 mark comparison dates for the above three cases; and k derives its meaning from formula (3.22).

1. The equation of value at the end of the 14th year is

$$100\, s_{\overline{14}|} + X_1 = 1000(1.05)^{14}.$$

Thus,

$$\begin{aligned} X_1 &= 1000(1.05)^{14} - 100\, s_{\overline{14}|} \\ &= 1979.93 - 1959.86 \\ &= \$20.07. \end{aligned}$$

2. The equation of value at the end of the 15th year is

$$100\, \ddot{s}_{\overline{14}|} + X_2 = 1000(1.05)^{15}.$$

Thus,

$$\begin{aligned} X_2 &= 1000(1.05)^{15} - 100(s_{\overline{15}|} - 1) \\ &= 2078.93 - 2057.86 \\ &= \$21.07 . \end{aligned}$$

It should be noted that $20.07(1.05) = 21.07$, or that in general $X_1(1 + i) = X_2$. The reader should justify this result by general reasoning.

3. In this case the equation of value becomes

$$100\, a_{\overline{14+k}|} = 1000$$

or

$$a_{\overline{14+k}|} = 10$$

where $0 < k < 1$. This can be written as

$$\frac{1 - v^{14+k}}{i} = 10$$

or

$$v^{14+k} = 1 - 10i = .5 .$$

Thus,

$$(1.05)^{14+k} = 2$$

giving

$$\begin{aligned} 14 + k &= \frac{\log_e 2}{\log_e 1.05} \\ &= \frac{.693147}{.048790} \\ &= 14.2067 \end{aligned}$$

or

$$k = .2067 .$$

Thus, the exact final irregular payment from formula (3.22) is

$$X_3 = 100\,\frac{(1.05)^{.2067} - 1}{.05} = \$20.27$$

paid at time 14.2067. A common approximation to the exact amount would be $100k$ or \$20.67. The exact answer obtained lies between the answers to 1 and 2, as we would expect. It is obvious that using formula (3.22) is not only inconvenient but that it is also more difficult to use if exact answers are required.

Example 3.7 A fund of \$25,000 is to be accumulated by means of deposits of \$1000 made at the end of every year as long as necessary. If the fund earns an effective rate of interest of 8%, find how many regular deposits will be necessary and the size of a final deposit to be made one year after the last regular deposit.

The equation of value is

$$1000 s_{\overline{n}|} = 25{,}000$$

or

$$s_{\overline{n}|} = 25 .$$

By inspection of the interest tables, we have $14 < n < 15$.

Thus, it takes 14 regular deposits plus a smaller final deposit, X. The equation of value at the end of the 15th year is

$$1000 s_{\overline{14}|}^{*} + X = 25{,}000$$

Thus,

$$\begin{aligned} X &= 25{,}000 - 1000(s_{\overline{15}|} - 1) \\ &= 25{,}000 - 26{,}152 \\ &= -\$1152 \end{aligned}$$

to the nearest dollar.

What has happened here is that the last regular deposit brings the fund close enough to \$25,000 that interest alone over the last year is sufficient to cause the fund to exceed \$25,000. The balance in the fund at the end of the 14th year is

$$1000 s_{\overline{14}|} = \$24{,}215 .$$

The balance in the fund at the end of the 15th year with interest only over the last year is

$$24{,}215(1.08) = \$26{,}152$$

which is in excess of the desired fund by \$1152. The result agrees with the one above. This example should not be thought of as typical, since often a final deposit is necessary. However, it does illustrate that pitfalls do exist and that care must be used in obtaining reasonable answers.

3.8 UNKNOWN RATE OF INTEREST

In this section we consider the situation in which the rate of interest is the unknown. As noted in Section 2.7, problems involving the determination of an unknown rate of interest are widely encountered in practice.

We consider three methods to use in determining an unknown rate of interest. The first of these is to solve for i by algebraic techniques. For example, our fundamental definition for an n-year annuity-immediate

$$a_{\overline{n}|} = v + v^2 + \cdots + v^n \tag{3.1}$$

is an nth degree polynomial in v. If the roots of this polynomial can be determined algebraically, then i is immediately determined. This method is generally practical only for small values of n.

Alternatively, we can express $a_{\overline{n}|}$ or $1/a_{\overline{n}|}$ in terms of i and solve by algebraic techniques. As series expansions, we have

$$a_{\overline{n}|} = n - \frac{n(n+1)}{2!} i + \frac{n(n+1)(n+2)}{3!} i^2 - \cdots \tag{3.24}$$

and

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{n}\left[1 + \frac{n+1}{2}i + \frac{n^2-1}{12}i^2 + \cdots\right]. \tag{3.25}$$

The derivations of formulas (3.24) and (3.25) are left as exercises. Formula (3.25) is generally preferred to formula (3.24), since the rate of convergence is faster. The use of formula (3.24) or (3.25) is not a particularly attractive method of solving numerically for an unknown rate of interest. However, these formulas do have value in connection with the derivation of other more useful approximation formulas later in this section and in Chapter 7.

The second method is to use linear interpolation in the interest tables. This method is quite similar to that used in Section 2.7. It should be noted that the accuracy of linear interpolation will depend on how closely together the interest rates appearing in the tables are tabulated.

The third, and by far the best, method is *successive approximation* or *iteration*. This method can be applied to all problems of this type and can produce whatever level of accuracy is required by carrying out enough iterations. It does require the use of a calculator with exponential and logarithmic functions, whereas the first two methods above do not. The reader is referred to Appendix V for more background on iteration methods.

Iteration can easily be applied when an equation of the form

$$i = g(i) \tag{3.26}$$

exists and converges to the true value of i, which satisfies the formula exactly. Assume some starting value, labeled i_0. Then generate a value i_1 by

$$i_1 = g(i_0).$$

In general, $i_1 \neq i_0$, so that another trial will be necessary. Then generate a value i_2 by

$$i_2 = g(i_1).$$

If the iteration is converging, then the successive values $i_0, i_1, i_2, \ldots$ will converge to the true value i. In practice, iterations will be carried out until $i_{s+1} = i_s$ to the required degree of accuracy. The reader should note that the symbol i_s was used in Chapter 1 to represent the rate of interest during the sth period from the date of investment.

Consider the problem in which the value of $a_{\overline{n}|}$ is given as some constant k and it is desired to find the rate of interest i which will produce this value. One simple, straightforward iteration method can be obtained directly from formula (3.2)

$$i = \frac{1 - (1 + i)^{-n}}{k}. \tag{3.27}$$

Unfortunately, as we will see in Example 3.8, the rate of convergence for this iteration formula is quite slow.

A powerful method defined in Appendix V, which converges very rapidly, is the *Newton-Raphson iteration method.* The iteration formula for solving $a_{\overline{n}|i} = k$ by this method is

$$i_{s+1} = i_s \left[1 + \frac{1 - (1 + i_s)^{-n} - k i_s}{1 - (1 + i_s)^{-n-1}\{1 + i_s(n + 1)\}} \right]. \tag{3.28}$$

The deviation of formula (3.28) is left as an exercise. Formula (3.28) is a bit complicated, and may not be worth the trouble for isolated calculations. However it is an excellent method if a large number of calculations need to be performed, since it does have a very rapid rate of convergence.

Applying an iteration method requires the use of a starting value. The number of iterations will be greatly reduced if a starting value close to the true root is used. Good starting values can be obtained by linear interpolation in the interest tables, i.e. by the second method described above.

However, a more convenient method in determining starting values than interpolation in the interest tables is to apply directly an approximation formula based only on n and k. Such a formula can be derived from the first two terms of formula (3.25)

$$\frac{1}{k} = \frac{1}{a_{\overline{n}|}} \fallingdotseq \frac{1}{n}\left[1 + \frac{n + 1}{2} i\right]$$

which gives

$$\frac{n + 1}{2n} i \fallingdotseq \frac{1}{k} - \frac{1}{n} = \frac{n - k}{nk}$$

or

$$i \fallingdotseq \frac{2(n - k)}{k(n + 1)}. \tag{3.29}$$

Formula (3.29) is analogous to a commonly-used formula for computing approximate bond yields given in Section 7.6. A verbal interpretation of formula (3.29) will be given at that time.

Analogous results to formulas (3.28) and (3.29) can be derived for accumulated values. The Newton-Raphson iteration formula for solving $s_{\overline{n}|i} = k$ is

$$i_{s+1} = i_s \left[1 + \frac{(1 + i_s)^n - 1 - k i_s}{(1 + i_s)^{n-1}\{1 - i_s(n-1)\} - 1} \right]. \tag{3.30}$$

The formula analogous to (3.29) is given by

$$i \doteq \frac{2(k - n)}{k(n - 1)}. \tag{3.31}$$

The derivations of formulas (3.30) and (3.31) are left as exercises.

Appendix V contains other iteration algorithms not illustrated in this section which can also be applied to solve for unknown rates of interest.

Example 3.8 At what rate of interest, convertible quarterly, is $16,000 the present value of $1000 paid at the end of every quarter for five years?

Let $j = i^{(4)}/4$, so that the equation of value becomes

$$1000\, a_{\overline{20}|j} = 16{,}000$$

or

$$a_{\overline{20}|j} = 16.$$

We will first illustrate interpolation in the interest tables. Define

$$f(j) = a_{\overline{20}|j} - 16.$$

We seek to find j, such that $f(j) = 0$. By inspection of the interest tables

$$f(.0200) = 16.3514 - 16 = .3514$$

and

$$f(.0250) = 15.5892 - 16 = -.4108$$

Now performing a linear interpolation

$$j = .0200 + .0050 \frac{.3514 + 0}{.3514 + .4108} = .0223$$

which gives

$$i^{(4)} = 4(.0223) = .0892, \quad \text{or} \quad 8.92\%.$$

We will next illustrate iteration. For a starting value we could use the value derived by linear interpolation above, i.e. $j_0 = .0223$. However, we will instead apply formula (3.29) to obtain

$$\frac{2(20 - 16)}{(16)(21)} = .0238.$$

We first illustrate an iteration using formula (3.27). We obtain the following results:

$$\begin{array}{llll} j_0 = .0238 & j_4 = .02283 & j_8 = .02248 & j_{12} = .02234 \\ j_1 = .02345 & j_5 = .02270 & j_9 = .02243 & j_{13} = .02233 \\ j_2 = .02319 & j_6 = .02261 & j_{10} = .02239 & j_{14} = .02231 \\ j_3 = .02298 & j_7 = .02253 & j_{11} = .02237 & j_{15} = .02230 \end{array}$$

We stop after 15 cycles, since the iteration is converging so slowly. We illustrate such a slow iteration, not because it is a good method to use, but rather to demonstrate the care which must be taken in practice to use iteration methods with a reasonable rate convergence.

We now try the Newton-Raphson iteration method using formula (3.28). We obtain the following results:

$$\begin{aligned} j_0 &= .0238 \\ j_1 &= .0222459 \\ j_2 &= .0222623 \\ j_3 &= .0222623 \end{aligned}$$

We have achieved seven decimal places of accuracy after only two iterations! The slow ad hoc method given by formula (3.27) could not even achieve five decimal places of accuracy after 15 iterations. The power of the Newton-Raphson method is indeed evident.

Actually, a much faster ad hoc iteration than that exhibited by formula (3.27) can be achieved by using the technique illustrated in Example 2.10. Using this approach requires two steps in each cycle, but will achieve one additional decimal place of accuracy per cycle. The reader will be asked to perform such an iteration in the exercises.

The correct answer to six decimal places is thus

$$i^{(4)} = 4(.0222623) = .089049, \quad \text{or} \quad 8.9049\% .$$

Example 3.9 Derive the following alternative formula for a starting value to solve $a_{\overline{n}|i} = k$ and apply it to Example 3.8:

$$i_0 = \frac{1 - \left[\frac{k}{n}\right]^2}{k}. \tag{3.32}$$

Consider n quantities: $v, v^2, v^3, \ldots, v^n$. The arithmetic mean of these quantities is

$$\frac{v + v^2 + v^3 + \cdots + v^n}{n} = \frac{a_{\overline{n}|}}{n} = \frac{k}{n}.$$

The geometric mean of these quantities is

$$v^{\frac{1+2+3+\cdots+n}{n}} = v^{\frac{n+1}{2}}.$$

However, we know that the arithmetic mean of n positive numbers, not all of which are equal, is greater than the geometric mean, and thus we have

$$\frac{k}{n} > v^{\frac{n+1}{2}}.$$

We replace the right-hand side by a slightly larger number and assume that

$$\frac{k}{n} \doteq v^{\frac{n}{2}}$$

or

$$(1+i)^{-n} \doteq \left[\frac{k}{n}\right]^2.$$

Formula (3.32) is now obtained immediately by substitution into formula (3.27).

Applying formula (3.32) to Example 3.8 produces

$$j_0 = \frac{1 - \left[\frac{16}{20}\right]^2}{16} = .0225$$

which produces a starting value closer to the true value than formula (3.29).

It can be shown that an analogous formula for a starting value to solve $s_{\overline{n}|i} = k$ is given by

$$i_0 = \frac{\left[\frac{k}{n}\right]^2 - 1}{k}. \tag{3.33}$$

A third formula for starting values, which also produces excellent results, can be obtained as a by-product of material convered in Section 8.4 and will be given at that time.

3.9 VARYING INTEREST

Thus far we have assumed a level rate of interest throughout the term of the annuity. In this section we will consider the situation in which the rate of interest can vary each period, but compound interest is still in effect. Other patterns of variation, not involving compound interest, are considered in Section 3.10.

As in Chapter 1, let i_k denote the rate of interest applicable for period k, i.e. for the interval from time $k-1$ to time k. We consider first the present value of an n-period annuity-immediate.

Two patterns of variation could be involved. The first pattern would be for i_k to be the applicable rate for period k regardless of when the payment is made. In this case the present value becomes

$$a_{\overline{n}|} = (1+i_1)^{-1} + (1+i_1)^{-1}(1+i_2)^{-1} + \cdots + (1+i_1)^{-1}(1+i_2)^{-1}\cdots(1+i_n)^{-1} = \sum_{t=1}^{n}\prod_{s=1}^{t}(1+i_s)^{-1}. \quad (3.34)$$

The second pattern would be to compute present values using rate i_k for the payment made at time k over all k periods. In this case the present value becomes

$$a_{\overline{n}|} = (1+i_1)^{-1} + (1+i_2)^{-2} + \cdots + (1+i_n)^{-n} = \sum_{t=1}^{n}(1+i_t)^{-t}. \quad (3.35)$$

We now turn to accumulated values. We will consider values of $\ddot{s}_{\overline{n}|}$ rather than $s_{\overline{n}|}$ so that all values of i_k for $k = 1, 2, \ldots, n$ will enter the formula. Again, the same two patterns of variation could be involved. If the applicable rate for period k is i_k regardless of when the payment is made, we have

$$\ddot{s}_{\overline{n}|} = (1+i_n) + (1+i_n)(1+i_{n-1}) + \cdots + (1+i_n)(1+i_{n-1})\ldots(1+i_1) = \sum_{t=1}^{n}\prod_{s=1}^{t}(1+i_{n-s+1}). \quad (3.36)$$

Alternatively, if the payment made at time k earns at rate i_k over the rest of the accumulation period, we have

$$\ddot{s}_{\overline{n}|} = (1+i_n) + (1+i_{n-1})^2 + \cdots + (1+i_1)^n = \sum_{t=1}^{n}(1+i_{n-t+1})^t. \quad (3.37)$$

Accumulated values of the annuity-immediate can be obtained from accumulated values of the annuity-due by using formula (3.16), i.e. $s_{\overline{n+1}|} = \ddot{s}_{\overline{n}|} + 1$.

The reader should distinguish carefully between the two different patterns of computing interest described above, since both occur in practice. In essence, formulas (3.34) and (3.36) assume that the interest rate for any given period is the same for all payments whose value is affected by interest during that period. On the other hand, formulas (3.35) and (3.37) assume that each payment has an associated interest rate which remains level over the entire period for which present values or accumulated values are being computed. The former case will be seen to have an important practical application in Section 10.2, while the latter case will be seen to have an important practical application in Section 9.6.

In practice, changes in interest rates often do not occur each period, but only once every several periods. In such cases, values can be obtained directly

from annuity values given earlier in this chapter. This approach is generally easier to use tnan to directly apply the formulas given above.

Example 3.10 Find the accumulated value of a 10-year annuity-immediate of $100 per year if the effective rate of interest is 5% for the first 6 years and 4% for the last 4 years.

We could apply formula (3.36); however, the following approach is simpler. The accumulated value of the first six payments after six years is

$$100 s_{\overline{6}|.05}.$$

This value is accumulated to the end of the 10 years at 4%, giving

$$100 s_{\overline{6}|.05}(1.04)^4.$$

The accumulated value of the last four payments is

$$100 s_{\overline{4}|.04}.$$

Thus, the answer is

$$\begin{aligned} & 100\left[s_{\overline{6}|.05}(1.04)^4 + s_{\overline{4}|.04}\right] \\ = \; & 100[(6.8019)(1.16986) + 4.2465 \\ = \; & \$1220.38 \end{aligned}$$

Example 3.11 Rework Example 3.10 if the first 6 payments are invested at an effective rate of interest of 5% and if the final 4 payments are invested at 4%.

We could apply formula (3.37), but again a simpler approach is available. The set-up is similar to Example 3.10 and the answer is

$$\begin{aligned} & 100\left[s_{\overline{6}|.05}(1.05)^4 + s_{\overline{4}|.04}\right] \\ = \; & 100[(6.8019)(1.21551) + 4.2465 \\ = \; & \$1251.43 . \end{aligned}$$

The reader should justify the relative magnitude of the answers to Examples 3.10 and 3.11 by general reasoning.

3.10 ANNUITIES NOT INVOLVING COMPOUND INTEREST

The valuation of annuities not involving compound interest is full of pitfalls and requires careful analysis and interpretation to obtain reasonable results. Multiple values for annuities can result, which in turn requires qualification as to the basis on which the values are calculated. In fact, it is best to avoid dealing with annuities not involving compound interest, if possible.

Unfortunately, it is not always possible to avoid the subject. For example, on occasion some courts have required the use of simple interest in computing the value of an annuity involving lost income in personal injury and wrongful death lawsuits. The "correct" procedure to compute such values is not only

ambiguous, but serious distortions will result if a significant period of time is involved in the calculation.

Also, the subject of annuity values not involving compound interest has appeared in various forms in the literature. Thus, the reader should be exposed to some of the results which have appeared and the ambiguities involved.

It would seem logical to return to the accumulation function as the basis to compute annuity values. The present value of an n-period annuity-immediate is equal to the sum of the present values of the individual payments. Thus, a generalized version of formula (3.1) is

$$a_{\overline{n}|} = \sum_{t=1}^{n} \frac{1}{a(t)} . \tag{3.38}$$

We turn next to finding a value for $s_{\overline{n}|}$. If we assume that 1 invested at time t, where $t = 1, 2, \ldots n-1$ will accumulate to $\frac{a(n)}{a(t)}$ at time n, then we would have

$$s_{\overline{n}|} = \sum_{t=1}^{n} \frac{a(n)}{a(t)} = a(n) \sum_{t=1}^{n} \frac{1}{a(t)} . \tag{3.39}$$

This would seem to be an appropriate procedure in certain cases, e.g. a problem involving a varying force of interest over the n periods. However, it does not produce correct results in all cases. For example, suppose we wish to find the accumulated value of an n-period annuity-immediate in which each payment is invested at simple interest from the date of payment until the end of the n periods. The accumulated value of such an annuity would be

$$1 + (1 + i) + (1 + 2i) + \cdots + [1 + (n-1)\,i] .$$

This leads to a generalized version of formula (3.3)

$$s_{\overline{n}|} = \sum_{t=0}^{n-1} a(t) \tag{3.40}$$

which will produce the correct answer for the above simple interest example.

However, it can easily be shown that formulas (3.39) and (3.40) do not produce the same result under simple interest. In fact, it can be shown that formulas (3.39) and (3.40) do not produce the same results in general for any pattern of interest other than compound interest.

For those readers who like symmetry and completeness, we could use formula (3.40) to find another expression for $a_{\overline{n}|}$

$$a_{\overline{n}|} = \sum_{t=0}^{n-1} \frac{a(t)}{a(n)} = \frac{1}{a(n)} \sum_{t=0}^{n-1} a(t) . \tag{3.41}$$

It is not surprising that formulas (3.38) and (3.41) produce different answers unless compound interest is involved.

Further analysis of annuity values at simple interest is instructive. Consider the investment of a sum of money for n periods at simple interest which is just sufficient to permit a withdrawal of 1 the end of each period for n periods. The concept of simple interest means that interest does not earn additional interest. Thus, the original amount of the investment is deposited in a fund which earns interest at rate i, but any interest earned is immediately transferred into a second fund which does not earn interest.

The ambiguity arises whenever a withdrawal is made. How much should be withdrawn from the principal fund, which earns interest, and how much from the interest fund, which does not earn interest? Different answers will be obtained depending on how this allocation is made.

In order to illustrate some of the possible answers which can be obtained, let $n = 6$ and $i = 10\%$. Let the original investment be denoted by K. We will refer to the principal fund as "Fund 1," and the interest fund as "Fund 2."

One extreme case would be to withdraw payments from Fund 2 whenever possible, and to withdraw from Fund 1 only when Fund 2 is exhausted. At the end of the first period interest of $.1K$ is earned on Fund 1 and deposited into Fund 2. A withdrawal of 1 is then made, with $.1K$ coming from Fund 2 and $1 - .1K$ coming from Fund 1. This leaves a new balance in Fund 1 of $K - (1 - .1K) = 1.1K - 1$. At the end of the second period interest of $.1(1.1K - 1)$ is earned on Fund 1 and deposited into Fund 2. A withdrawal of 1 is then made, with $.1[1.1K - 1]$ coming from Fund 2 and $1 - .1(1.1K - 1)$ coming from Fund 1. This leaves a new balance in Fund 1 of

$$\begin{aligned} &(1.1K - 1) - [1 - .1(1.1K - 1)] \\ &= (1.1)^2K - (1 + 1.1)\,. \end{aligned}$$

If this process is continued for four more years, the balance in Fund 1 at the end of six years is

$$(1.1)^6K - [1 + 1.1 + (1.1)^2 + (1.1)^3 + (1.1)^4 + (1.1)^5]$$

and the balance in Fund 2 is zero. However, the balance in Fund 1 must also be zero after the final withdrawal. Thus

$$K = \frac{(1.1)^6 - 1}{(1.1)^6(1.1 - 1)} = \frac{1 - (1.1)^{-6}}{.1} = 4.36$$

which is the value of $a_{\overline{6}|.1}$ using compound interest. We will discover in Chapter 6 that this is no accident. The results of this approach are summarized in Table 3.1.

Table 3.1 Fund Development - Withdrawals from Fund 2 Prior to Fund 1 for $a_{\overline{6}|}$ at 10%.

	Fund 1 Balance		*Fund 2 Balance*	
Period	*Before withdrawal*	*After withdrawal*	*Before withdrawal*	*After withdrawal*
0	N/A	4.355	N/A	0
1	4.355	3.791	.436	0
2	3.791	3.170	.379	0
3	3.170	2.487	.317	0
4	2.487	1.736	.249	0
5	1.736	.910	.174	0
6	.910	.001*	.091	0

* zero, except for .001 roundoff error

The other extreme case would be to withdraw payments from Fund 1 whenever possible, and to withdraw from Fund 2 only when Fund 1 is exhausted. In view of the answer above, we will assume that Fund 1 becomes exhausted at the end of the fifth period. It will be necessary to confirm that the answer obtained is consistent with this assumption. At the end of the first period interest of $.1K$ is earned on Fund 1 and deposited into Fund 2. A withdrawal of 1 is then made with all of it coming from Fund 1. This leaves a new balance of $K - 1$ in Fund 1 and $.1K$ in Fund 2. At the end of the second period interest of $.1(K - 1)$ is earned on Fund 1 and deposited into Fund 2. A withdrawal of 1 is then made with all of it coming from Fund 1. This leaves a new balance of $K - 2$ in Fund 1 and $.1K + .1(K - 1) = .2K - .1$ in Fund 2. If this process is continued for two more periods, we have $K - 4$ in Fund 1, which we have assumed lies between 0 and 1. Also, we have

$$.1[K + (K - 1) + (K - 2) + (K - 3)] = .4K - .6$$

in Fund 2. At the end of the fifth period interest of $.1(K - 4)$ is earned on Fund 1 and deposited into Fund 2. A withdrawal of 1 is then made with $K - 4$ of it coming from Fund 1, which is now exhausted. The rest of the payment, i.e. $1 - (K - 4) = 5 - K$, comes from Fund 2. The new balance in Fund 2 is

$$(.4K - .6) + .1(K - 4) - (5 - K) = 1.5K - 6 .$$

At the end of the sixth period no interest is earned, since Fund 1 is exhausted. The fund balance in Fund 2 is unchanged, since Fund 2 does not earn interest. The final withdrawal of 1 comes entirely from Fund 2 exhausting it, so that we have

$$1.5K - 6 = 1$$
$$K = \frac{7}{1.5} = 4.67 = a_{\overline{6}|}.$$

This answer is valid, since the assumption was made that $4 < K < 5$. If an answer had been obtained outside this interval, the problem would have to be reworked until an answer consistent with the assumption about the exhaustion of Fund 1 is obtained. The results of this approach are summarized in Table 3.2.

Table 3.2 Fund Development - Withdrawals from Fund 1 Prior to Fund 2 for $a_{\overline{6}|}$ at 10%

	Fund 1 Balance		*Fund 2 Balance*	
Period	*Before withdrawal*	*After withdrawal*	*Before withdrawal*	*After withdrawal*
0	N/A	4.667	N/A	0
1	4.667	3.667	.467	.467
2	3.667	2.667	.834	.834
3	2.667	1.667	1.101	1.101
4	1.667	.667	1.268	1.268
5	.667	0	1.268	1.002**
6	0	0	1.002	.002*

* zero, except for .002 roundoff error
** calculated as 1.268 + .1(.667) - (1 - .667) = 1.002

We next apply formula (3.38) directly and obtain

$$a_{\overline{6}|} = \frac{1}{1.1} + \frac{1}{1.2} + \cdots + \frac{1}{1.6} = 4.52$$

which lies between our two extreme answers already obtained. Apparently, formula (3.38) involves an implicit intermediate allocation of withdrawals from Fund 1 and Fund 2 between the two extremes.

If we apply formula (3.41), we obtain

$$a_{\overline{6}|} = \frac{1 + 1.1 + \cdots + 1.5}{1.6} = 4.69$$

which lies outside the range bounded by the two extremes. Thus, we must question whether formula (3.41) produces a result with any real-world significance at all, at least in this example. The results of this analysis are summarized in Table 3.3.

Table 3.3 Comparison of $a_{\overline{6}|}$ Computed at 10% Simple Interest on Four Different Bases

Basis	*Value*
Low extreme (compound interest)	4.36
Formula (3.38)	4.52
High extreme	4.67
Formula (3.41)	4.69

At this point the reader should be thoroughly convinced that finding annuity values at simple interest is treacherous indeed. Furthermore, simple discount would present just as many difficulties. The reader now should really appreciate the admonition given early in this section to avoid computing annuity values not involving compound interest if at all possible!

Example 3.12 *Compare the value of* $s_{\overline{6}|.1}$ *at 10% interest (1) assuming compound interest, (2) using formula (3.39), and (3) using formula (3.40).*

1. Using compound interest at 10%

$$s_{\overline{6}|} = 7.72 .$$

2. From formula (3.39)

$$s_{\overline{6}|} = 1.6 \left[\frac{1}{1.1} + \frac{1}{1.2} + \cdots + \frac{1}{1.6} \right] = 7.23 .$$

3. From formula (3.40)

$$s_{\overline{6}|} = 1 + 1.1 + 1.2 + \cdots + 1.5 = 7.50 .$$

Answer (3) involves simple interest being earned on each payment from the date of deposit to the end of the six-year period of investment. Answer 1 is larger than either of the other two, verifying the larger accumulation with compound interest.

Example 3.13 *Find* $s_{\overline{5}|}$ *, if* $\delta_t = .02t$ *for* $0 \le t \le 5$.

If we accumulate each payment from the date of deposit to the end of the five-year period at the varying force of interest δ_t, we have

$$\begin{aligned} s_{\overline{5}|} &= \sum_{t=1}^{5} e^{\int_t^5 \delta_r dr} = \sum_{t=1}^{5} e^{.01r^2 \big]_t^5} \\ &= e^{.24} + e^{.21} + e^{.16} + e^{.09} + 1 \\ &= 1.27125 + 1.23368 + 1.17351 + 1.09417 + 1 \\ &= 5.7726 . \end{aligned}$$

Alternatively, formula (3.39) is applicable in this case. We have

$$\begin{aligned} a(t) &= e^{\int_0^t \delta_r dr} \\ &= e^{.01r^2]_0^t} \\ &= e^{.01t^2}, \end{aligned}$$

so that

$$\begin{aligned} s_{\overline{5}|} &= a(5) \sum_{t=1}^{5} \frac{1}{a(t)} \\ &= e^{.25}\left(e^{-.01} + e^{-.04} + e^{-.09} + e^{-.16} + e^{-.25}\right) \\ &= e^{.24} + e^{.21} + e^{.16} + e^{.09} + 1 \\ &= 5.7726 . \end{aligned}$$

which agrees with the answer obtained from first principles above.

EXERCISES

3.2 Annuity-immediate

1. A family wishes to accumulate \$50,000 in a college education fund at the end of 20 years. If they deposit \$1000 in the fund at the end of each of the first 10 years and \$1000 + X in the fund at the end of each of the second 10 years, find X to the nearest dollar if the fund earns 7% effective.

2. The cash price of a new automobile is \$10,000. The purchaser is willing to finance the car at 18% convertible monthly and to make payments of \$250 at the end of each month for four years. Find the down payment which will be necessary.

3. An annuity provides a payment of n at the end of each year for n years. The annual effective interest rate is $1/n$. What is the present value of the annuity?

4. If $a_{\overline{n}|} = x$ and $a_{\overline{2n}|} = y$, express d as a function of x and y.

5. *a*) Show that $a_{\overline{m+n}|} = a_{\overline{m}|} + v^m a_{\overline{n}|} = v^n a_{\overline{m}|} + a_{\overline{n}|}$.
   b) Show that $s_{\overline{m+n}|} = s_{\overline{m}|} + (1+i)^m s_{\overline{n}|} = (1+i)^n s_{\overline{m}|} + s_{\overline{n}|}$.
   c) Verbally interpret the results in (*a*) and (*b*).

6. *a*) Show that $a_{\overline{m-n}|} = a_{\overline{m}|} - v^m s_{\overline{n}|} = (1+i)^n a_{\overline{m}|} - s_{\overline{n}|}$ where $0 < n < m$.
   b) Show that $s_{\overline{m-n}|} = s_{\overline{m}|} - (1+i)^m a_{\overline{n}|} = v^n s_{\overline{m}|} - a_{\overline{n}|}$ where $0 < n < m$.
   c) Verbally interpret the results in (*a*) and (*b*).

7. You are given the following annuity values:

$$a_{\overline{7}|} = 5.153 \quad a_{\overline{11}|} = 7.036 \quad a_{\overline{18}|} = 9.180$$

   Find i.

8. Show that $\dfrac{1}{1 - v^{10}} = \dfrac{1}{s_{\overline{10}|}} \left[s_{\overline{10}|} + \dfrac{1}{i} \right]$.

3.3 Annuity-due

9. Find the present value of payments of \$200 every six months starting immediately and continuing through four years from the present, and \$100 every six months thereafter through ten years from the present, if $i^{(2)} = .06$.

10. A worker aged 40 wishes to accumulate a fund for retirement by depositing \$1000 at the beginning of each year for 25 years. Starting at age 65 the worker plans to make 15 annual withdrawals at the beginning of each year. Assuming that all payments are certain to be made, find the amount of each withdrawal starting at age 65 to the nearest dollar, if the effective rate of interest is 8% during the first 25 years but only 7% thereafter.

11. Find $\ddot{a}_{\overline{8}|}$ if the effective rate of discount is 10%.

12. Derive formula (3.12).

13. *a*) Show that $\ddot{a}_{\overline{n}|} = a_{\overline{n}|} + 1 - v^n$.
    b) Show that $\ddot{s}_{\overline{n}|} = s_{\overline{n}|} - 1 + (1+i)^n$.
    c) Verbally interpret the results in (*a*) and (*b*).

14. Show that $\dfrac{\ddot{s}_{\overline{2n}|}}{\ddot{s}_{\overline{n}|}} + \dfrac{\ddot{s}_{\overline{n}|}}{\ddot{s}_{\overline{2n}|}} - \dfrac{\ddot{s}_{\overline{3n}|}}{\ddot{s}_{\overline{2n}|}} = 1.$

15. If $\ddot{a}_{\overline{p}|} = x$ and $s_{\overline{q}|} = y$, show that $a_{\overline{p+q}|} = \dfrac{vx + y}{1 + iy}$.

3.4 Annuity values on any date

16. Derive formulas (3.17), (3.18), and (3.19) algebraically.

17. Payments of \$100 per quarter are made from June 7, Z through December 7, $Z + 11$, inclusive. If the nominal rate of interest convertible quarterly is 6%:
    a) Find the present value on September 7, $Z - 1$.
    b) Find the current value on March 7, $Z + 8$.
    c) Find the accumulated value on June 7, $Z + 12$.

18. Show that $\displaystyle\sum_{t=10}^{15} (\ddot{s}_{\overline{t}|} - s_{\overline{t}|}) = s_{\overline{16}|} - s_{\overline{10}|} - 6.$

19. Annuities X and Y provide the following payments:

End of Year	Annuity X	Annuity Y
1-10	1	K
11-20	2	0
21-30	1	K

Annuities X and Y have equal present values at an annual effective interest rate i such that $v^{10} = 1/2$. Determine K.

20. At an annual effective interest rate i it is known that:
    (*i*) The present value of 2 at the end of each year for $2n$ years, plus an additional 1 at the end of each of the first n years, is 36.
    (*ii*) The present value of an n-year deferred annuity-immediate paying 2 per year for n years is 6.
    Find i.

21. It is known that $\dfrac{a_{\overline{7}|}}{a_{\overline{11}|}} = \dfrac{a_{\overline{3}|} + s_{\overline{x}|}}{a_{\overline{y}|} + s_{\overline{z}|}}$. Find x, y, and z.

22. Simplify $a_{\overline{15}|}(1 + v^{15} + v^{30})$ to one symbol.

23. Find the present value to the nearest dollar on January 1 of an annuity which pays \$2000 every six months for five years. The first payment is due on the next April 1 and the rate of interest is 9% convertible semiannually.

3.5 Perpetuities

24. A sum P is used to buy a deferred perpetuity-due of 1 payable annually. The annual effective rate of interest is $i > 0$. Find an expression for the deferred period.

25. Deposits of \$1000 are placed into a fund at the beginning of each year for the next 20 years. After 30 years annual payments commence and continue forever, with the first payment at the end of the 30th year. Find an expression for the amount of each payment.

26. A benefactor leaves an inheritance to four charities, A, B, C, and D. The total inheritance is a series of level payments at the end of each year forever. During the first n years A, B, and C share each payment equally. All payments after n years revert to D. If the present values of the shares of A, B, C, and D are all equal, find $(1 + i)^n$.

27. A level perpetuity-immediate is to be shared by A, B, C, and D. A receives the first n payments, B the second n payments, C the third n payments, and D all payments thereafter. It is known that the ratio of the present value of C's share to A's share is .49. Find the ratio of the present value of B's share to D's share.

3.6 Nonstandard terms and interest rates

28. Find an expression for the error involved in approximating $\dfrac{(1 + i)^k - 1}{i}$ by k.

29. Compute $a_{\overline{5.25}|}$ if $i = 5\%$ using the following definitions:
    a) Formula (3.22).
    b) A payment of .25 at time 5.25.
    c) A payment of .25 at time 6.

a) Formula (3.22).
b) A payment of .25 at time 5.25.
c) A payment of .25 at time 6.

30. In the text annuity symbols were defined only for positive values of n. Find expressions consistent with formulas (3.2) and (3.4) for:
a) $a_{\overline{-n}|}$.
b) $s_{\overline{-n}|}$.

31. Derive the following formulas for annuity values computed at a negative rate of interest:
a) $\ddot{s}_{\overline{n}|i} = a_{\overline{n}|-d}$.
b) $\ddot{a}_{\overline{n}|i} = s_{\overline{n}|-d}$.

3.7 Unknown time

32. A loan of \$1000 is to be repaid by annual payments of \$100 to commence at the end of the fifth year and to continue thereafter for as long as necessary. Find the time and amount of the final payment, if the final payment is to be larger than the regular payments. Assume $i = 4\ 1/2\%$.

33. A fund of \$2000 is to be accumulated by n annual payments of \$50, followed by n annual payments of \$100, plus a smaller final payment made one year after the last regular payment. If the effective rate of interest is 4 1/2%, find n and the amount of the final irregular payment.

34. One annuity pays 4 at the end of each year for 36 years. Another annuity pays 5 at the end of each year for 18 years. The present values of both annuities are equal at effective rate of interest i. If an amount of money invested at the same rate i will double in n years, find n.

35. A fund earning 8% effective is being accumulated with payments of \$500 at the beginning of each year for 20 years. Find the maximum number of withdrawals of \$1000 which can be made at the ends of years under the condition that once withdrawals start they must continue through the end of the 20-year period.

36. A borrower has the following two options for repaying a loan:
(*i*) Sixty monthly payments of \$100 at the end of each month.
(*ii*) A single payment of \$6000 at the end of K months.
Interest is at the nominal annual rate of 12% convertible monthly. The two options have the same present value. Find K.

3.8 Unknown rate of interest

37. Derive the following:

38. If $a_{\overline{2}|} = 1.75$, find an exact expression for i.

39. Rework Example 3.8 using the ad hoc iteration technique illustrated in Example 2.10.

40. Rework Example 3.8 using the iteration formula

$$j_{s+2} = \frac{j_s f(j_{s+1}) - j_{s+1} f(j_s)}{f(j_{s+1}) - f(j_s)}$$

with the two starting values obtained in Example 3.8, i.e. let $j_0 = .0200$ and $j_1 = .0250$.

41. A fund of \$17,000 is to be accumulated at the end of five years with payments at the end of each half-year. The first five payments are \$1000 each, while the second five payments are \$2000 each. Find the nominal rate of interest convertible semiannually earned on the fund.

42. A beneficiary receives a \$10,000 life insurance benefit. If the beneficiary uses the proceeds to buy a 10-year annuity-immediate, the annual payout will be \$1538. If a 20-year annuity-immediate is purchased, the annual payout will be \$1072. Both calculations are based on an annual effective interest rate of i. Find i.

3.9 Varying interest

43. *a*) Find the present value of an annuity-immediate which pays 1 at the end of each half-year for five years, if the rate of interest is 8% convertible semiannually for the first three years and 7% convertible semiannually for the last two years.
    b) Find the present value of an annuity-immediate which pays 1 at the end of each half-year for five years, if the payments for the first three years are discounted at 8% convertible semiannually and the payments for the last two years are discounted at 7% convertible semiannually.
    c) Justify from general reasoning that the answer to (*b*) is larger than the answer to (*a*).

44. A loan of P is to be repaid by 10 annual payments beginning 6 months from the date of the loan. The first payment is to be half as large as the others. For the first 4 1/2 years interest is i effective; for the remainder of the term interest is j effective. Find an expression for the first payment.

45. You are given:
    (*i*) X is the current value at time 2 of a 20-year annuity-due of 1 per annum.
    (*ii*) The annual effective interest rate for year t is $\frac{1}{8+t}$.

    Find X.

(*ii*) The annual effective interest rate for year t is $\dfrac{1}{8+t}$.

Find X.

3.10 Annuities not involving compound interest

46. Find an expression for $a_{\overline{n}|}$ assuming each payment is valued at simple discount rate d.

47. If $a(t) = \dfrac{1}{\log_2(t+2) - \log_2(t+1)}$, find an expression for $\ddot{a}_{\overline{n}|}$ by directly taking the present value of the payments.

48. Given that $\delta_t = \dfrac{1}{20-t}$, $t \geq 0$, find $s_{\overline{10}|}$.

49. For time $t > 0$, the discount function is defined by

$$a^{-1}(t) = \frac{1}{1 + .01t}.$$

A five-year annuity has payments of 1 at times $t = 1, 2, 3, 4, 5$. A calculates the present value of this annuity at time 0 directly. However, B first accumulates the payments according to the accumulation function

$$a(t) = 1 + .01t.$$

B then multiplies the result by $a^{-1}(5)$. By how much do the answers of A and B differ?

Miscellaneous problems

50. *a*) Show that $s_{\overline{n}|} = n + \dfrac{n(n-1)}{2!}\,i + \dfrac{n(n-1)(n-2)}{3!}\,i^2 + \cdots$.

b) Show that $\dfrac{1}{s_{\overline{n}|}} = \dfrac{1}{n}\left[1 - \dfrac{n-1}{2}\,i + \dfrac{n^2-1}{12}\,i^2 - \cdots\right]$.

51. A loan of \$1000 is to be repaid with annual payments at the end of each year for the next 20 years. For the first 5 years the payments are k per year; the second 5 years, $2k$ per year; the third 5 years, $3k$ per year; and the fourth 5 years, $4k$ per year. Find an expression for k.

52. The present value of an annuity-immediate which pays \$200 every 6 months during the next 10 years and \$100 every 6 months during the following 10 years is \$4000. The present value of a 10-year deferred annuity-immediate which pays \$250 every 6 months for 10 years is \$2500. Find the present value of an annuity-immediate which pays \$200 every 6 months during the next 10 years and \$300 every 6 months during the following 10 years. (Hint: Payments made during the first 10 years are discounted at a different rate than payments made during the second 10 years.)

53. A depositor puts \$10,000 into a bank account that pays an annual effective interest rate of 4% for 10 years. If a withdrawal is made during the first 5 1/2 years, a penalty of 5% of the withdrawal amount is made. The depositor withdraws K at the end of each of years 4, 5, 6, and 7. The balance in the account at the end of year 10 is \$10,000. Find K.

54. Simplify $\sum_{n=15}^{40} s_{\overline{n}|}$.

55. Show that $s_{\overline{n}|}\ddot{a}_{\overline{n}|} > n^2$ if $i > 0$ and $n > 1$.

4

More general annuities

4.1 INTRODUCTION

In Chapter 3 we discussed annuities for which the payment period and the interest conversion period are equal and coincide, and for which the payments are of level amount. In Chapter 4 annuities for which payments are made more or less frequently than interest is convertible and annuities with varying payments will be considered.

4.2 ANNUITIES PAYABLE AT A DIFFERENT FREQUENCY THAN INTEREST IS CONVERTIBLE

We first address annuities for which the payment period and the interest conversion period differ and for which the payments are level. There are two distinct approaches that can be followed in handling such annuities.

The first approach is applicable if the only objective is to compute the numerical value of an annuity and a calculator with exponential and logarithmic functions is available. In this case a two-step procedure can be followed:

1. Find the rate of interest, convertible at the same frequency as payments are made, which is equivalent to the given rate of interest.
2. Using this new rate of interest, find the value of the annuity using the techniques discussed in Chapter 3.

This approach is general and can be used for annuities payable more or less frequently than interest is convertible. Morever, unknown time and unknown rate of interest problems can be handled in this fashion. This direct approach will be illustrated in the examples.

The second approach involves an algebraic analysis of such annuities. The intent is to develop algebraic expressions for such annuities in terms of annuity symbols already defined in Chapter 3, with adjustment factors sometimes being required.

The reader might expect that the algebraic development for annuities payable less frequently than interest is convertible would be quite similar to that for annuities payable more frequently than interest is convertible. Surprisingly, however, the two cases have traditionally been developed in the literature with definitions and formulas which look quite different from each other. The algebraic analysis of annuities payable less frequently than interest is convertible is given in Section 4.3, while Section 4.4 analyzes annuities payable more frequently than interest is convertible.

Although these algebraic approaches are not required if the sole purpose is to find numerical values for such annuities, they do provide valuable analytical insight into annuities in general. Also, they provide an important foundation for the analysis of contingent annuities (e.g. a monthly annuity payable for life from a pension plan valued at an effective rate of interest). Finally, they offer an alternative computational approach in which interest tables can be utilized.

***Example 4.1** Find the accumulated value at the end of four years of an investment fund in which \$100 is deposited at the beginning of each quarter for the first two years and \$200 is deposited at the beginning of each quarter for the second two years, if the fund earns 12% convertible quarterly.*

We are given an interest rate of 1% per month. Let j be the equivalent rate of interest per quarter, which is the payment period. We have

$$j = (1.01)^3 - 1 = .030301.$$

The value of the annuity in symbols is

$$100\,(\ddot{s}_{\overline{16}|j} + \ddot{s}_{\overline{8}|j})$$

which can be evaluted as

$$100(20.8170 + 9.1716) = \$2999$$

to the nearest dollar.

***Example 4.2** A loan of \$3000 is to be repaid with quarterly installments at the end of each quarter for five years. If the rate of interest charged on the loan is 10% convertible semiannually, find the amount of each quarterly payment.*

We are given an interest rate of 5% per half-year. Let j be the equivalent rate of interest per quarter, which is the payment period. We have

$$j = (1.05)^{1/2} - 1 = .024695.$$

Let the quarterly payment be denoted by R. Then the equation of value is

$$R\,a_{\overline{20}|j} = 3000$$

so that

$$R = \frac{3000}{a_{\overline{20}|j}} = \frac{3000}{15.6342} = \$191.89.$$

Example 4.3 At what annual effective rate of interest will payments of $100 at the end of every quarter accumulate to $2500 at the end of five years?

Let $j = i^{(4)}/4$ be the interest rate per quarter which accomplishes the above. Then the equation of value at the end of five years is

$$100\,s_{\overline{20}|j} = 2500$$

or

$$s_{\overline{20}|j} = 25.$$

We use formula (3.33) to obtain a starting value for the iteration

$$i_0 = \frac{\left[\frac{25}{20}\right]^2 - 1}{25} = .0225.$$

We next use the Newton-Raphson method of iteration. Using formula (3.30) we generate the following successive values:

$$i_1 = .022855$$
$$i_2 = .022854$$
$$i_3 = .022854$$

The annual effective rate of interest i is given by

$$i = (1.022854)^4 - 1 = .0946, \text{ or } 9.46\%.$$

4.3 FURTHER ANALYSIS OF ANNUITIES PAYABLE LESS FREQUENTLY THAN INTEREST IS CONVERTIBLE

In this section annuities payable less frequently than interest is convertible are further analyzed algebraically. This section will be subdivided into the following areas: (1) annuity-immediate, (2) annuity-due, and (3) other considerations.

Annuity-immediate

Let k be the number of interest conversion periods in one payment period, let n be the term of the annuity measured in interest conversion periods, and let i be the rate of interest per interest conversion period. We will assume that each payment period contains an integral number of interest conversion periods;

thus k and n are both positive integers. The number of annuity payments made is n/k, which is also a positive integer.

The present value of an annuity which pays 1 at the end of each k interest conversion periods for a total of n interest conversion periods is

$$\begin{aligned} v^k + v^{2k} + \cdots + v^{\frac{n}{k} \cdot k} &= \frac{v^k - v^{n+k}}{1 - v^k} \\ &= \frac{1 - v^n}{(1+i)^k - 1} \\ &= \frac{a_{\overline{n}|}}{s_{\overline{k}|}} . \end{aligned} \tag{4.1}$$

Thus, we have an expression for the present value of this annuity in terms of annuity symbols already defined.

The accumulated value of this annuity immediately after the last payment is

$$\frac{a_{\overline{n}|}}{s_{\overline{k}|}}(1+i)^n = \frac{s_{\overline{n}|}}{s_{\overline{k}|}} . \tag{4.2}$$

It is possible to derive formulas (4.1) and (4.2) by an alternate argument. There is a value of R such that the series of payments of 1 at the end of each k interest conversion periods for n interest conversion periods can be replaced by a series of payments of R at the end of each interest conversion period so that the present values are equal. The present value of this series is

$$Ra_{\overline{n}|} .$$

Now consider any one payment period which contains k interest conversion periods. At the end of the payment period the accumulated value of payments of R at the end of each interest conversion period must equal the payment of 1 made at that point. Thus,

$$Rs_{\overline{k}|} = 1$$

and substituting $R = 1/s_{\overline{k}|}$ into $Ra_{\overline{n}|}$, formula (4.1) is obtained. Formula (4.2) can be derived by a similar argument.

Figure 4.1. is a time diagram clarifying the above argument.

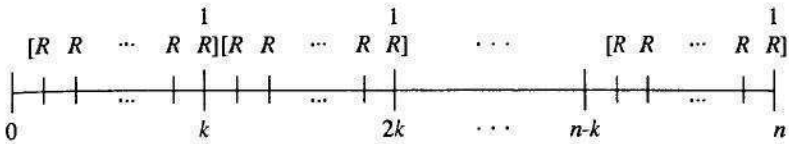

Figure 4.1 Time diagram for an annuity-immediate payable less frequently than interest is convertible

Annuity-due

The present value of an annuity which pays 1 at the beginning of each k interest conversion periods for a total of n interest conversion periods is

$$1 + v^k + v^{2k} + \cdots + v^{n-k} = \frac{1 - v^n}{1 - v^k} = \frac{a_{\overline{n}|}}{a_{\overline{k}|}} . \tag{4.3}$$

The accumulated value of this annuity k interest conversion periods after the last payment is

$$\frac{a_{\overline{n}|}}{a_{\overline{k}|}} (1 + i)^n = \frac{s_{\overline{n}|}}{a_{\overline{k}|}} . \tag{4.4}$$

It is possible to derive formulas (4.3) and (4.4) by an alternate argument analogous to the argument used above for the annuity-immediate. The reader should fill in the details for this argument. Figure 4.2 is a time diagram for this case.

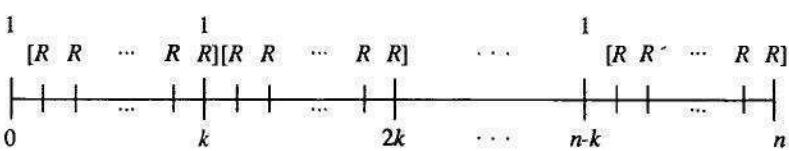

Figure 4.2 Time diagram for an annuity-due payable less frequently than interest is convertible

Other considerations

On occasion a perpetuity payable less frequently than interest is convertible is encountered. The present value of such a perpetuity-immediate is

$$v^k + v^{2k} + \cdots = \frac{v^k}{1 - v^k}$$
$$= \frac{1}{(1+i)^k - 1}$$
$$= \frac{1}{i\, s_{\overline{k}|}} \tag{4.5}$$

which is also the limit of formula (4.1) as n approaches infinity. Similarly, the present value of a perpetuity-due is

$$\frac{1}{i\, a_{\overline{k}|}}. \tag{4.6}$$

A second special case occasionally encountered is to find the value of a series of payments at a given force of interest δ. Although coming under the category of annuities payable less frequently than interest is convertible, this situation is not adequately handled by the methods discussed above, since n and k are both infinite. This situation can best be handled by writing an expression for the value of the annuity as the sum of present values or accumulated values of each separate payment, replacing v^{tk} with $e^{-\delta tk}$ and $(1+i)^{tk}$ with $e^{\delta tk}$. This expression can be summed as a geometric progression. An illustration of this type appears in the exercises.

A third special case, very rarely encountered in practice, is a situation in which each payment period does not contain an integral number of interest conversion periods (i.e. $k > 1$, but k is not integral). Here again, the best approach would be to resort to basic principles, i.e. to write an expression as the sum of present values or accumulated values of each separate payment, and then to sum this expression as a geometric progression. An illustration of this type also appears in the exercises.

It should be observed that no one-symbol expressions for the annuity values given by formulas (4.1) through (4.6) have been defined. The only expressions are those in terms of ordinary annuity symbols given by formulas (4.1) through (4.6).

Finally, it is possible to generalize the approach to finding annuity values on any date, as discussed in Section 3.4, to annuities payable less frequently than interest is convertible. Example 4.4 illustrates this approach.

***Example 4.4** Find an expression for the present value of an annuity in which there are a total of r payments of 1, the first to be made at the end of seven years, and the remaining payments at three-year intervals, at an annual effective rate i, expressed as: (1) an annuity-immediate, and (2) an annuity-due.*

Figure 4.3 is a time diagram for this example.

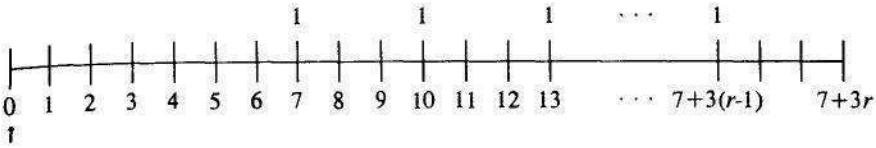

Figure 4.3 Time diagram for Example 4.4

The present value of this annuity is given by

$$v^7 + v^{10} + v^{13} + \cdots + v^{3r+4}.$$

1. Summing the geometric progression, we have

$$\frac{v^7 - v^{3r+7}}{1 - v^3} = \frac{v^4 - v^{3r+4}}{(1+i)^3 - 1} = \frac{(1 - v^{3r+4}) - (1 - v^4)}{(1+i)^3 - 1} = \frac{a_{\overline{3r+4}|} - a_{\overline{4}|}}{s_{\overline{3}|}}.$$

   Note that the annuity-immediate form is characterized by the $s_{\overline{3}|}$ in the denominator.

2. Summing the geometric progression, we have

$$\frac{v^7 - v^{3r+7}}{1 - v^3} = \frac{(1 - v^{3r+7}) - (1 - v^7)}{1 - v^3} = \frac{a_{\overline{3r+7}|} - a_{\overline{7}|}}{a_{\overline{3}|}}.$$

   Note that the annuity-due form is characterized by the $a_{\overline{3}|}$ in the denominator.

With practice the reader should be able to directly write down the answer to problems of this type in either the annuity-immediate form or the annuity-due form without actually summing the geometric progression.

Example 4.5 Rework Example 4.1 using the approach developed in Section 4.3.

The rate of interest is 1% per month, the term of the annuity is 48 interest conversion periods, and each payment period contains three interest conversion periods. Since this is an annuity-due, the accumulated value is

$$100\,\frac{s_{\overline{48}|.01} + s_{\overline{24}|.01}}{a_{\overline{3}|.01}} = 100\,\frac{61.2226 + 26.9735}{2.9410} = \$2999$$

using the interest tables and rounding to the nearest dollar. The answer agrees with that obtained in Example 4.1.

***Example 4.6** An investment of \$1000 is used to make payments of \$100 at the end of each year for as long as possible with a smaller final payment to be made at the time of the last regular payment. If interest is 7% convertible semiannually, find the number of payments and the amount of the total final payment.*

The equation of value is

$$100\frac{a_{\overline{n}|.035}}{s_{\overline{2}|.035}} = 1000$$

or

$$a_{\overline{n}|.035} = 10s_{\overline{2}|.035} = 20.35.$$

By inspection of the interest tables, we have $36 < n < 37$. Thus, 18 regular payments and a smaller final payment can be made. Let the smaller additional payment at the time of the final regular payment be denoted by R. Then an equation of value at the end of 18 years is

$$R + 100\frac{s_{\overline{36}|.035}}{s_{\overline{2}|.035}} = 1000(1.035)^{36}$$

or

$$R = 1000(3.45027) - 100\frac{70.0076}{2.0350} = \$10.09.$$

The total final payment would thus be \$110.09.

4.4 FURTHER ANALYSIS OF ANNUITIES PAYABLE MORE FREQUENTLY THAN INTEREST IS CONVERTIBLE

In this section annuities payable more frequently than interest is convertible are further analyzed algebraically. In practice annuities payable more frequently than interest is convertible are more common than annuities payable less frequently than interest is convertible. This section will be subdivided into the following areas: (1) annuity-immediate, (2) annuity-due, and (3) other considerations.

Annuity-immediate

Let m be the number of payment periods in one interest conversion period, let n be the term of the annuity measured in interest conversion periods, and let i be the interest rate per interest conversion period. We will assume that each interest conversion period contains an integral number of payment periods; thus m and n are both positive integers. The number of annuity payments made is mn, which is also a positive integer.

The present value of an annuity which pays $1/m$ at the end of each mth of an interest conversion period for a total of n interest conversion periods is denoted by $a_{\overline{n}|}^{(m)}$. We have

$$a_{\overline{n}|}^{(m)} = \frac{1}{m}\left[v^{\frac{1}{m}} + v^{\frac{2}{m}} + \cdots + v^{n-\frac{1}{m}} + v^{n}\right]$$

$$= \frac{1}{m}\left[\frac{v^{\frac{1}{m}} - v^{n+\frac{1}{m}}}{1 - v^{\frac{1}{m}}}\right]$$

$$= \frac{1 - v^{n}}{m\left[(1+i)^{\frac{1}{m}} - 1\right]}$$

$$= \frac{1 - v^{n}}{i^{(m)}}. \tag{4.7}$$

The accumulated value of this annuity immediately after the last payment is made is denoted by $s_{\overline{n}|}^{(m)}$, and we have

$$s_{\overline{n}|}^{(m)} = a_{\overline{n}|}^{(m)}(1+i)^{n}$$

$$= \frac{(1+i)^{n} - 1}{i^{(m)}}. \tag{4.8}$$

Formulas (4.7) and (4.8) should be compared with formulas (3.2) and (3.4), respectively. They are identical except that the denominators of (4.7) and (4.8) are $i^{(m)}$ instead of i. Since $i^{(m)}$ is a measure of interest paid at the end of $m$ths of an interest conversion period, the points at which interest is paid under this measure are consistent with the points at which payments are made. This property, relating the manner in which payments are made and the measure of interest in the denominator, was originally mentioned in Section 3.3.

It is possible to write $a_{\overline{n}|}^{(m)}$ and $s_{\overline{n}|}^{(m)}$ in terms of $a_{\overline{n}|}$ and $s_{\overline{n}|}$ with an adjustment factor. The following are immediate consequences of formulas (4.7) and (4.8):

$$a_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}}\, a_{\overline{n}|} \tag{4.9}$$

and

$$s_{\overline{n}|}^{(m)} = \frac{i}{i^{(m)}}\, s_{\overline{n}|}. \tag{4.10}$$

Formulas (4.9) and (4.10) are commonly applied in obtaining numerical results when interest tables are used. Values of $i/i^{(m)}$ appear in the interest tables in Appendix I. The term $i/i^{(m)}$ is often written as $s_{\overline{1}|}^{(m)}$, which is consistent with formula (4.8), setting $n = 1$.

Annuity-due

The present value of an annuity which pays $1/m$ at the beginning of each mth of an interest conversion period for a total of n interest conversion periods is denoted by $\ddot{a}_{\overline{n}|}^{(m)}$. We have

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{1 - v^n}{d^{(m)}} . \tag{4.11}$$

The derivation of formula (4.11) is similar to formula (4.7) and is left as an exercise.

The accumulated value of this annuity one mth of an interest conversion period after the last payment is made is denoted by $\ddot{s}_{\overline{n}|}^{(m)}$, and we have

$$\begin{aligned} \ddot{s}_{\overline{n}|}^{(m)} &= \ddot{a}_{\overline{n}|}^{(m)} (1 + i)^n \\ &= \frac{(1 + i)^n - 1}{d^{(m)}} . \end{aligned} \tag{4.12}$$

Here, again, the relationship between the manner in which payments are made and the measure of interest in the denominator should be noted.

It is an immediate consequence of formulas (4.11) and (4.12) that

$$\ddot{a}_{\overline{n}|}^{(m)} = \frac{i}{d^{(m)}} a_{\overline{n}|} \tag{4.13}$$

and

$$\ddot{s}_{\overline{n}|}^{(m)} = \frac{i}{d^{(m)}} s_{\overline{n}|} . \tag{4.14}$$

Formulas (4.13) and (4.14) can be used in obtaining numerical results if values of $i/d^{(m)} = \ddot{s}_{\overline{1}|}^{(m)}$ are available in the interest tables. Such values do appear in Appendix I, but in many interest tables they do not.

However, it is possible to develop alternative formulas which can be applied. Since each payment under $\ddot{a}_{\overline{n}|}^{(m)}$ is made one mth of an interest conversion period earlier than $a_{\overline{n}|}^{(m)}$, we have

$$\begin{aligned} \ddot{a}_{\overline{n}|}^{(m)} &= (1 + i)^{\frac{1}{m}} a_{\overline{n}|}^{(m)} \\ &= \left(1 + \frac{i^{(m)}}{m} \right) \frac{i}{i^{(m)}} a_{\overline{n}|} \\ &= \left(\frac{i}{i^{(m)}} + \frac{i}{m} \right) a_{\overline{n}|} \end{aligned} \tag{4.15}$$

and similarly,

$$\ddot{s}_{\overline{n}|}^{(m)} = \left(\frac{i}{i^{(m)}} + \frac{i}{m} \right) s_{\overline{n}|} \,. \tag{4.16}$$

Formulas (4.15) and (4.16) are convenient in numerical calculations using interest tables when values of $i/i^{(m)}$ are available and values of $i/d^{(m)}$ are not.

A number of other interesting identities involving annuities payable $m$thly can be developed. In particular, identities analogous to formulas (3.6) and (3.12) through (3.16) exist. These six identities and their derivations are left as exercises.

Other considerations

On occasion a perpetuity payable more frequently than interest is convertible is encountered. The following formulas are analogous to formulas (3.20) and (3.21):

$$a_{\overline{\infty}|}^{(m)} = \frac{1}{i^{(m)}} \tag{4.17}$$

and

$$\ddot{a}_{\overline{\infty}|}^{(m)} = \frac{1}{d^{(m)}}. \tag{4.18}$$

A second special case, very rarely encountered in practice, is a situation in which each interest conversion period does not contain an integral number of payment periods (i.e. $m > 1$, but m is not integral). In this case, the best approach would be to resort to basic principles, i.e. to write an expression as the sum of present values or accumulated values of each separate payment, and then to sum this expression as a geometric progression. An illustration of this type appears in the exercises.

One consideration which is important in practice involves the proper coefficients for annuities payable $m$thly. Each payment made is of amount $1/m$, while the coefficient of the symbol is 1. In general, the proper coefficient is the amount paid during one interest conversion period, and not the amount of each actual payment. The amount paid during one interest conversion period is often called the *annual rent* of the annuity. This is an appropriate term if the interest conversion period is one year, as is often the case; but it is confusing if the interest conversion period is other than one year. Sometimes the term *periodic rent* is used instead of "annual rent" to avoid this confusion.

Finally, it is possible to generalize the approach to finding annuity values on any date, as discussed in Section 3.4, to annuities payable more frequently than interest is convertible. Example 4.7 illustrates this approach.

Example 4.7 Payments of $400 per month are made over a ten-year period. Find expressions for: (1) the present value of these payments two years prior to the first payment, and (2) the accumulated value three years after the final payment. Use symbols based on an effective rate of interest.

1. The answer is

$$4800\, v^2\, \ddot{a}_{\overline{10}|}^{(12)} = 4800\left(\ddot{a}_{\overline{12}|}^{(12)} - \ddot{a}_{\overline{2}|}^{(12)}\right).$$

2. The answer is

$$4800\, s_{\overline{10}|}^{(12)}\,(1+i)^3 = 4800\left(s_{\overline{13}|}^{(12)} - s_{\overline{3}|}^{(12)}\right).$$

Example 4.8 Rework Example 4.2 using the approach developed in Section 4.4.

The rate of interest is 5% per half-year, the term of the loan is ten interest conversion periods, and each interest conversion period contains two payment periods. Thus, the equation of value is

$$2Ra_{\overline{10}|\,.05}^{(2)} = 3000$$

so that

$$R = \frac{1500}{a_{\overline{10}|\,.05}^{(2)}} = \frac{1500}{\frac{i}{i^{(2)}}\, a_{\overline{10}|\,.05}}$$

$$= \frac{1500}{(1.012348)(7.7217)} = \$191.89.$$

The answer agrees with that obtained in Example 4.2.

Example 4.9 At what annual effective rate of interest is the present value of a series of payments of $1 every six months forever, with the first payment made immediately, equal to $10?

The equation of value is

$$10 = 1 + v^{.5} + v + v^{1.5} + \cdots = \frac{1}{1 - v^{.5}}.$$

Thus,

$$v^{.5} = .9$$

and

$$(1+i)^{.5} = \frac{1}{.9}$$

which gives

$$i = \left[\frac{1}{.9}\right]^2 - 1 = .2346 \text{ or } 23.46\%.$$

4.5 CONTINUOUS ANNUITIES

A special case of annuities payable more frequently than interest is convertible is one in which the frequency of payment becomes infinite, i.e. payments are made continuously. Although difficult to visualize in practice, a continuous annuity is of considerable theoretical and analytical significance. Also, it is useful as an approximation to annuities payable with great frequency, such as daily.

We will denote the present value of an annuity payable continuously for n interest conversion periods, such that the total amount paid during each interest conversion period is 1, by the symbol $\bar{a}_{\overline{n}|}$. An expression for $\bar{a}_{\overline{n}|}$ is

$$\bar{a}_{\overline{n}|} = \int_0^n v^t dt \tag{4.19}$$

since the differential expression $v^t dt$ is the present value of the payment dt made at exact moment t.

A simplified expression can be obtained by performing the integration

$$\begin{aligned} \bar{a}_{\overline{n}|} &= \int_0^n v^t dt \\ &= \frac{v^t}{\log_e v}\Bigg]_0^n \\ &= \frac{1 - v^n}{\delta}. \end{aligned} \tag{4.20}$$

Formula (4.20) is analogous to formula (3.2). Again, there is consistency between the manner in which payments are made and the denominator of the expression.

Also formula (4.20) could have been obtained as follows:

$$\bar{a}_{\overline{n}|} = \lim_{m\to\infty} a^{(m)}_{\overline{n}|} = \lim_{m\to\infty} \frac{1 - v^n}{i^{(m)}} = \frac{1 - v^n}{\delta}$$

or

$$\bar{a}_{\overline{n}|} = \lim_{m\to\infty} \ddot{a}^{(m)}_{\overline{n}|} = \lim_{m\to\infty} \frac{(1 - v^n)}{d^{(m)}} = \frac{1 - v^n}{\delta}.$$

Thus, the continuous annuity is seen to be the limiting case of annuities payable $m$thly.

It is possible to write $\bar{a}_{\overline{n}|}$ in terms of $a_{\overline{n}|}$ with an adjustment factor

$$\bar{a}_{\overline{n}|} = \frac{i}{\delta} a_{\overline{n}|} = \bar{s}_{\overline{1}|}\, a_{\overline{n}|}. \tag{4.21}$$

Values of $i/\delta = \bar{s}_{\overline{1}|}$ can be readily calculated directly and also appear in the interest tables in Appendix I.

The accumulated value of a continuous annuity at the end of the term of the annuity is denoted by $\bar{s}_{\overline{n}|}$. The following relationships hold:

$$\bar{s}_{\overline{n}|} = \int_0^n (1+i)^t \, dt \tag{4.22}$$

$$= \frac{(1+i)^t}{\log_e(1+i)} \Bigg]_0^n$$

$$= \frac{(1+i)^n - 1}{\delta} \tag{4.23}$$

$$= \frac{i}{\delta} s_{\overline{n}|} = \bar{s}_{\overline{1}|} \, s_{\overline{n}|} \tag{4.24}$$

$$= \lim_{m\to\infty} s^{(m)}_{\overline{n}|} = \lim_{m\to\infty} \ddot{s}^{(m)}_{\overline{n}|} .$$

Additional insight into continuous annuities can be obtained by differentiating formula (4.22) with respect to its upper limit n, and then replacing n with t, which gives

$$\frac{d}{dt}\bar{s}_{\overline{t}|} = (1+i)^t$$

$$= 1 + \delta \bar{s}_{\overline{t}|} \tag{4.25}$$

upon substituting formula (4.23). Formula (4.25) has an interesting verbal interpretation. Consider an investment fund in which money is continuously being deposited at the rate of 1 per interest conversion period. The fund balance at time t is equal to $\bar{s}_{\overline{t}|}$. The fund balance is changing instantaneously for two reasons. First, new deposits are occurring at the constant rate of 1 per interest conversion period, and second, interest is being earned at force δ on the fund balance $\bar{s}_{\overline{t}|}$.

Similarly, by differentiating formula (4.19) we have

$$\frac{d}{dt}\bar{a}_{\overline{t}|} = v^t$$

$$= 1 - \delta \bar{a}_{\overline{t}|} . \tag{4.26}$$

Formula (4.26) also has a verbal interpretation, which is deferred until some additional material is presented in Chapter 6.

We can express the value of continuous annuities strictly in terms of the force of interest δ. When this is done, formula (4.20) becomes

$$\bar{a}_{\overline{n}|} = \frac{1 - e^{-n\delta}}{\delta} \tag{4.27}$$

and formula (4.23) becomes

$$\bar{s}_{\overline{n}|} = \frac{e^{n\delta} - 1}{\delta} . \tag{4.28}$$

In a sense, this is a case of an annuity in which the payment period and interest conversion period are equal, but which was not considered in Chapter 3.

Example 4.10 Find the force of interest at which $\bar{s}_{\overline{20}|} = 3\,\bar{s}_{\overline{10}|}$.

Using formula (4.28) we have

$$\frac{e^{20\delta} - 1}{\delta} = 3\,\frac{e^{10\delta} - 1}{\delta}$$

or

$$e^{20\delta} - 3e^{10\delta} + 2 = 0$$

which factors into

$$(e^{10\delta} - 2)(e^{10\delta} - 1) = 0.$$

However, $e^{10\delta} - 1 = 0$ implies that $\delta = 0$, which clearly is an extraneous root. Thus, we have

$$e^{10\delta} = 2$$

so that

$$\delta = \frac{\log_e 2}{10} = .0693, \text{ or } 6.93\%.$$

4.6 BASIC VARYING ANNUITIES

Thus far all the annuities considered have had level payments. We now remove this restriction and consider annuities with varying payments. In Section 4.6 it will be assumed that the payment period and interest conversion period are equal and coincide. As in Chapter 3, we will just use the term "period" for both.

Naturally, any type of *varying annuity* can be evaluated by taking the present value or the accumulated value of each payment separately and summing the results. On occasion, this may be the only feasible approach. However, there are several types of varying annuities for which relatively simple expressions can be developed, and we will consider these.

The following types of varying annuities will be discussed in this section: (1) payments varying in arithmetic progression, (2) payments varying in geometric progression, and (3) other payment patterns.

Payments varying in arithmetic progression

Consider a general annuity-immediate with a term of n periods in which payments begin at P and increase by Q per period thereafter. The interest rate is i per period. Figure 4.4. is a time diagram for this annuity. It should be

noted that P must be positive but that Q can be either positive or negative as long as $P + (n-1)Q > 0$ to avoid negative payments.

Let A be the present value of the annuity. Then we have

$$A = Pv + (P+Q)v^2 + (P+2Q)v^3 + \cdots + [P+(n-2)Q]v^{n-1} + [P+(n-1)Q]v^n.$$

This series is a combination of an arithmetic and a geometric progression. Such a series can be solved algebraically by multiplying by the common ratio in the geometric progression to give

$$(1+i)A = P + (P+Q)v + (P+2Q)v^2 + (P+3Q)v^3 + \cdots + [P+(n-1)Q]v^{n-1}.$$

Now subtracting the first equation from the second

$$\begin{aligned} iA &= P + Q(v + v^2 + v^3 + \cdots + v^{n-1}) - Pv^n - (n-1)Qv^n \\ &= P(1-v^n) + Q(v + v^2 + v^3 + \cdots + v^{n-1} + v^n) - Qnv^n. \end{aligned}$$

Thus,

$$\begin{aligned} A &= P\frac{1-v^n}{i} + Q\frac{a_{\overline{n}|} - nv^n}{i} \\ &= Pa_{\overline{n}|} + Q\frac{a_{\overline{n}|} - nv^n}{i}. \end{aligned} \tag{4.29}$$

The accumulated value is given by

$$Ps_{\overline{n}|} + Q\frac{s_{\overline{n}|} - n}{i} \tag{4.30}$$

since it must be the present value accumulated for n periods.

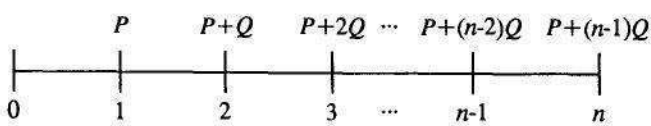

Figure 4.4 Time diagram for formulas (4.29) and (4.30)

Formulas (4.29) and (4.30) can be used in solving any problem in which payments vary in arithmetic progression. However, there are two special cases which often appear and have special notation.

The first of these is the *increasing annuity* in which $P = 1$ and $Q = 1$. Figure 4.5 is a time diagram for this annuity. The present value of this annuity, denoted by $(Ia)_{\overline{n}|}$, can be obtained from formula (4.29)

$$\begin{aligned}(Ia)_{\overline{n}|} &= a_{\overline{n}|} + \frac{a_{\overline{n}|} - nv^n}{i} \\ &= \frac{1 - v^n + a_{\overline{n}|} - nv^n}{i} \\ &= \frac{\ddot{a}_{\overline{n+1}|} - (n+1)v^n}{i} \\ &= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}. \end{aligned} \tag{4.31}$$

The accumulated value of this annuity, denoted by $(Is)_{\overline{n}|}$, is

$$\begin{aligned}(Is)_{\overline{n}|} &= (Ia)_{\overline{n}|}\ (1+i)^n \\ &= \frac{\ddot{s}_{\overline{n}|} - n}{i} = \frac{s_{\overline{n+1}|} - (n+1)}{i}. \end{aligned} \tag{4.32}$$

Formula (4.31) can be derived by an alternative approach which considers an increasing annuity to be the summation of a series of level deferred annuities. Applying formula (3.17), we have

$$\begin{aligned}(Ia)_{\overline{n}|} &= \sum_{t=0}^{n-1} v^t a_{\overline{n-t}|} \\ &= \sum_{t=0}^{n-1} v^t \cdot \frac{1 - v^{n-t}}{i} \\ &= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}. \end{aligned} \tag{4.31}$$

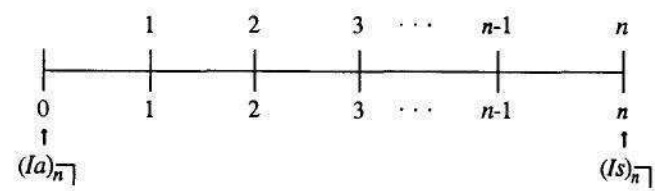

Figure 4.5 Time diagram for an increasing annuity

Not only is this derivation quite efficient, but it also provides valuable insight into the nature of an increasing annuity.

The second of these is the *decreasing annuity* in which $P = n$ and $Q = -1$. Figure 4.6 is a time diagram for this annuity. The present value of this annuity, denoted by $(Da)_{\overline{n}|}$, can be obtained from formula (4.29)

$$\begin{aligned}(Da)_{\overline{n}|} &= na_{\overline{n}|} - \frac{a_{\overline{n}|} - nv^n}{i} \\ &= \frac{n - nv^n - a_{\overline{n}|} + nv^n}{i} \\ &= \frac{n - a_{\overline{n}|}}{i}. \end{aligned} \tag{4.33}$$

The accumulated value of this annuity, denoted by $(Ds)_{\overline{n}|}$, is

$$\begin{aligned}(Ds)_{\overline{n}|} &= (Da)_{\overline{n}|}\,(1+i)^n \\ &= \frac{n(1+i)^n - s_{\overline{n}|}}{i}. \end{aligned} \tag{4.34}$$

Formula (4.33) can also be derived by the alternative approach which considers a decreasing annuity to be the summation of a series of level annuities. Following this approach, we have

$$\begin{aligned}(Da)_{\overline{n}|} &= \sum_{t=1}^{n} a_{\overline{t}|} \\ &= \sum_{t=1}^{n} \frac{1 - v^t}{i} \\ &= \frac{n - a_{\overline{n}|}}{i}. \end{aligned} \tag{4.33}$$

Again, the alternative derivation is quite efficient and provides valuable insight into the nature of a decreasing annuity.

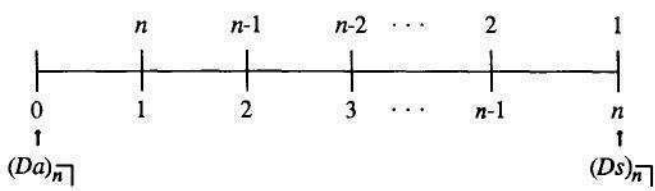

Figure 4.6 Time diagram for a decreasing annuity

All the above formulas are for annuities-immediate. However, we may use the previously mentioned relationship between the manner in which payments are made and the denominator of the expression for the annuity value to find formulas for annuities-due. Changing i in the denominator of any of the above formulas to d will produce values for annuities-due.

Also, it is possible to have varying perpetuities. We can find the general form for a perpetuity by taking the limit of formula (4.29) as n approaches infinity, obtaining

$$\frac{P}{i} + \frac{Q}{i^2} \tag{4.35}$$

since

$$\lim_{n\to\infty} a_{\overline{n}|} = \frac{1}{i} \text{ and } \lim_{n\to\infty} nv^n = 0.$$

Note that P and Q must both be positive to avoid negative payments.

An alternative approach to finding expressions for varying annuities is to make use of the following three quantities:

$$F_n = v^n \tag{4.36}$$

$$= \text{The present value of a payment of 1 at the end of } n \text{ periods.}$$

$$G_n = \frac{v^n}{d} \tag{4.37}$$

$$= \text{The present value of a level perpetuity of 1 per period, first payment at the end of } n \text{ periods.}$$

$$H_n = \frac{v^n}{d^2} \tag{4.38}$$

$$= \text{The present value of an increasing perpetuity of } 1, 2, 3, \ldots, \text{ first payment at the end of } n \text{ periods.}$$

These symbols are very useful in setting up expressions for varying annuities. By appropriately describing the pattern of payments, expressions for the annuity can be immediately written down. This will be illustrated in the examples.

The formulas developed in this subsection have interesting verbal interpretations. For example, consider formula (4.31) written as

$$\ddot{a}_{\overline{n}|} = i(Ia)_{\overline{n}|} + nv^n.$$

The left-hand side represents the present value of an investment of 1 at the beginning of each period for n periods. The two terms on the right-hand side can be interpreted, respectively, as the present value of the interest to be earned and the present value of the return of the principal invested.

Payments varying in geometric progression

Annuities with payments varying in geometric progression can be readily handled by directly expressing the annuity value as a series with each payment multiplied by its associated present or accumulated value. Since the payments and the present or accumulated values are both geometric progressions, the terms in the series for the annuity value constitute a new geometric progression.

By way of example, consider an annuity-immediate with a term of n periods in which the first payment is 1 and successive payments increase in geometric progression with common ratio $1 + k$. The present value of this annuity is

$$v + v^2(1 + k) + \cdots + v^n(1 + k)^{n-1}.$$

However, this is a geometric progession whose sum is

$$v\left[\frac{1 - \left[\frac{1+k}{1+i}\right]^n}{1 - \left[\frac{1+k}{1+i}\right]}\right] = \frac{1 - \left[\frac{1+k}{1+i}\right]^n}{i - k}. \tag{4.39}$$

This expression can be evaluated by direct calculation. On occasion, either $\frac{1+k}{1+i}$ or $\frac{1+i}{1+k}$ may equal $1 + j$ for some j with tabular interest functions, in which case the interest tables can be used. If $k = i$, formula (4.39) is undefined. However, then the present value is just nv, which is obvious from the original series.

The present value of a perpetuity will exist if $0 < \frac{1+k}{1+i} < 1$, in which case the sum of the geometric progression exists. If $\frac{1+k}{1+i} \geq 1$, then the geometric progression diverges and the present value of the perpetuity does not exist.

Other payment patterns

In practical applications the value of varying annuities in which other payment patterns exist are generally best handled from first principles. The

annuity value can be computed by finding the present value or accumulated value of each payment separately and summing the results. Appendix VI contains a further analysis of varying annuities whose payments follow a polynomial of degree $n > 1$. These results are required in an application contained in Section 9.9.

The reader should also be careful to distinguish the term "varying annuity" from the term "variable annuity." A *variable annuity* is a type of life annuity in which payments vary according to the investment experience of an underlying investment account, usually one invested in common stocks. Variable annuities are beyond the scope of this book and are not considered further.

Example 4.11 Use the technique involving the functions F_n, G_n, and H_n to derive: (1) formula (4.31), and (2) formula (4.33).

1. The payments represented by $(Ia)_{\overline{n}|}$ can be described by

$$H_1 - H_{n+1} - n \cdot G_{n+1}$$

and substituting we obtain

$$\frac{v}{d^2} - \frac{v^{n+1}}{d^2} - n\frac{v^{n+1}}{d} = \frac{1}{di} - \frac{v^n}{di} - \frac{nv^n}{i}$$
$$= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}.$$

2. The payments represented by $(Da)_{\overline{n}|}$ can be described by

$$n \cdot G_1 - (H_2 - H_{n+2})$$

and substituting we obtain

$$n \cdot \frac{v}{d} - \left[\frac{v^2}{d^2} - \frac{v^{n+2}}{d^2}\right] = \frac{n}{i} - \frac{1}{i^2}(1 - v^n)$$
$$= \frac{n - a_{\overline{n}|}}{i}.$$

Example 4.12 Find the present value of a perpetuity-immediate whose successive payments are 1, 2, 3, 4, . . . , at an effective rate of interest of 5%.

An appropriate symbol would be $(Ia)_{\overline{\infty}|}$. Substituting $P = 1$ and $Q = 1$ into formula (4.35), we obtain

$$\frac{1}{i} + \frac{1}{i^2} = \frac{1}{.05} + \frac{1}{.0025} = 420.$$

Alternatively, we have

$$(Ia)_{\overline{\infty}|} = H_1 = \frac{v}{d^2} = \frac{1+i}{i^2} = \frac{1.05}{.0025} = 420.$$

Example 4.13 Find the present value of an annuity-immediate such that payments start at 1, increase by annual amounts of 1 to a payment of n, and then decrease by annual amounts of 1 to a final payment of 1.

The present value is

$$\begin{aligned}(Ia)_{\overline{n}|} + v^n(Da)_{\overline{n-1}|} &= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} + v^n\frac{(n-1) - a_{\overline{n-1}|}}{i} \\ &= \frac{1}{i}\left[a_{\overline{n-1}|} + 1 - nv^n + nv^n - v^n - v^n a_{\overline{n-1}|}\right] \\ &= \frac{1}{i}\left[a_{\overline{n-1}|}(1 - v^n) + (1 - v^n)\right] \\ &= \frac{1}{i}(1 - v^n)(a_{\overline{n-1}|} + 1) \\ &= a_{\overline{n}|} \cdot \ddot{a}_{\overline{n}|}.\end{aligned}$$

Alternatively, we can express the present value of the payments as

$$\begin{aligned}(H_1 - H_{n+1}) - (H_{n+1} - H_{2n+1}) &= H_1 - 2H_{n+1} + H_{2n+1} \\ &= \frac{v - 2v^{n+1} + v^{2n+1}}{d^2} \\ &= \frac{1 - 2v^n + v^{2n}}{id} \\ &= \frac{(1 - v^n)^2}{id} \\ &= a_{\overline{n}|} \cdot \ddot{a}_{\overline{n}|}.\end{aligned}$$

As an exercise, the reader will be asked to give a verbal interpretation to this answer.

Example 4.14 Find the present value of an annuity-immediate such that payments start at 1, each payment thereafter increases by 1 until reaching 10, and then remain at that level until 25 payments in total are made.

A symbol for an n-period increasing annuity-immediate in which increases are limited to the first m periods only, $0 < m < n$, is given by

$$(I_{\overline{m}|}a)_{\overline{n}|}.$$

Thus, the present value of the annuity in this example can be written as $(I_{\overline{10}|}a)_{\overline{25}|}$. The reader should verify that the following are all valid expressions for the present value of this annuity:

- $(Ia)_{\overline{10}|} + 10v^{10}a_{\overline{15}|}$
- $(Ia)_{\overline{25}|} - v^{10}(Ia)_{\overline{15}|}$
- $10a_{\overline{25}|} - (Da)_{\overline{9}|}$
- $\sum_{t=0}^{9} v^t a_{\overline{25-t}|}$
- $H_1 - H_{11} - 10G_{26}$

Example 4.15 An annuity provides for 20 annual payments, the first payment a year hence being $1000. The payments increase in such a way that each payment is 4% greater than the preceding payment. Find the present value of this annuity at an annual effective rate of interest of 7%.

Using formula (4.39) we have

$$1000\,\frac{1-\left[\frac{1.04}{1.07}\right]^{20}}{.07-.04}=\$14{,}459$$

to the nearest dollar.

4.7 MORE GENERAL VARYING ANNUITIES

The varying annuities considered in Section 4.6 assumed that the payment period and the interest conversion period are equal and coincide. In Section 4.7 this restriction is removed. In practice, varying annuities with payments made more or less frequently than interest is convertible occur infrequently.

We will consider generalizations of the increasing annuity, $(Ia)_{\overline{n}|}$, in which interest is convertible more or less frequently than payments are made. Other annuities in which payments vary in arithmetic progression can be handled analogously.

Consider first the case in which payments are made less frequently than interest is convertible. Let k be the number of interest conversion periods in one payment period, let n be the term of the annuity measured in interest conversion periods, and let i be the rate of interest per interest conversion period. The number of payments is n/k, which is integral.

Let A be the present value of the generalized increasing annuity. We have

$$A = v^k + 2v^{2k} + \cdots + \left[\frac{n}{k} - 1\right] v^{n-k} + \frac{n}{k} v^n$$

and

$$(1+i)^k A = 1 + 2v^k + \cdots + \left[\frac{n}{k} - 1\right] v^{n-2k} + \frac{n}{k} v^{n-k}.$$

Now subtracting the first equation from the second

$$A\left[(1+i)^k - 1\right] = 1 + v^k + v^{2k} + \cdots + v^{n-k} - \frac{n}{k} v^n$$

which can be expressed as

$$A = \frac{\frac{a_{\overline{n}|}}{a_{\overline{k}|}} - \frac{n}{k} v^n}{i s_{\overline{k}|}}. \tag{4.40}$$

Formula (4.40) is a generalized version of formula (4.31) and the reader should note the similarity.

Consider next the case in which payments are made more frequently than interest is convertible. Two different results arise depending on whether the rate of payment is constant or varies during each interest conversion period.

Consider first the situation in which the rate of payment is constant during each interest conversion period with increases occurring only once per interest conversion period. We can utilize the relationship between the manner in which payments are made and the measure of interest in the denominator to obtain the following generalized version of formula (4.31)

$$(Ia)^{(m)}_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i^{(m)}}. \tag{4.41}$$

Formula (4.41) gives the present value of n-period annuity-immediate, payable $m$thly, in which each payment during the first period is $1/m$, each payment during the second period is $2/m$, and so forth, until each payment during the nth period is n/m.

Consider next the situation in which the rate of payment changes with each payment period. Suppose that an increasing annuity is payable at the rate of $1/m$ per interest conversion period at the end of the first mth of an interest conversion period, $2/m$ per interest conversion period at the end of the second mth of an interest conversion period, and so forth. Then the first payment will be $1/m^2$, the second will be $2/m^2$, and so forth. Denoting the present value of such an annuity by $(I^{(m)}a)^{(m)}_{\overline{n}|}$, we have

$$\begin{aligned}(I^{(m)}a)^{(m)}_{\overline{n}|} &= \frac{1}{m^2}\left[v^{\frac{1}{m}} + 2v^{\frac{2}{m}} + \cdots + nmv^{\frac{nm}{m}}\right] \\ &= \frac{\ddot{a}^{(m)}_{\overline{n}|} - nv^n}{i^{(m)}}.\end{aligned} \tag{4.42}$$

The derivation of formula (4.42) is left as an exercise.

Annuities in which the payments vary in geometric progression and in which the payment period and the interest conversion period differ are occasionally encountered. However, such annuities present no new difficulties. They can be readily handled by expressing the annuity value as a summation of the present value or accumulated value of each payment. This summation is a geometric progression which can be directly evaluated. This technique will be illustrated in Example 4.17.

Example 4.16 Find the present value of a perpetuity which pays 1 at the end of the third year, 2 at the end of the sixth year, 3 at the end of the ninth year, . . .

Denote the present value of this perpetuity by A. Then

$$A = v^3 + 2v^6 + 3v^9 + \cdots$$

and

$$v^3A = \qquad v^6 + 2v^9 + \cdots$$

Now subtracting the second equation from the first

$$A(1 - v^3) = v^3 + v^6 + v^9 + \cdots = \frac{v^3}{1 - v^3}$$

or

$$A = \frac{v^3}{(1 - v^3)^2}.$$

Example 4.17 Find the accumulated value at the end of ten years of an annuity in which payments are made at the beginning of each half-year for five years. The first payment is $2000, and each payment is 98% of the prior payment. Interest is credited at 10% convertible quarterly.

We count periods in quarters of a year. The accumulated value is

$$\begin{aligned} &2000\left[(1.025)^{40} + (.98)(1.025)^{38} + (.98)^2(1.025)^{36} + \cdots + (.98)^9(1.025)^{22}\right] \\ &= 2000\,\frac{(1.025)^{40} - (.98)^{10}(1.025)^{20}}{1 - (.98)(1.025)^{-2}} \\ &= \$40{,}052 \end{aligned}$$

to the nearest dollar.

4.8 CONTINUOUS VARYING ANNUITIES

The last type of varying annuity we will consider is one in which payments are being made continuously at a varying rate. Such annuities are primarily of theoretical interest.

Consider an increasing annuity for n interest conversion periods in which payments are being made continuously at the rate of t per period at exact moment t. The present value of this annuity is denoted by $(\bar{Ia})_{\overline{n}|}$, and an expression for it would be

$$(\bar{Ia})_{\overline{n}|} = \int_0^n tv^t\,dt \tag{4.43}$$

since the differential expression $tv^t\,dt$ is the present value of the payment $t\,dt$ made at exact moment t.

A simplified expression can be obtained by performing an integration by parts. We have

$$\begin{aligned}(\bar{I}\bar{a})_{\overline{n}|} &= \int_0^n t v^t dt \\ &= \frac{t v^t}{\log_e v}\Bigg]_0^n - \int_0^n \frac{v^t}{\log_e v} dt \\ &= \frac{t v^t}{\log_e v}\Bigg]_0^n - \frac{v^t}{(\log_e v)^2}\Bigg]_0^n \\ &= -\frac{n v^n}{\delta} - \frac{v^n}{\delta^2} + \frac{1}{\delta^2} \\ &= \frac{1 - v^n}{\delta^2} - \frac{n v^n}{\delta} \\ &= \frac{\bar{a}_{\overline{n}|} - n v^n}{\delta}.\end{aligned} \tag{4.44}$$

It should be noted that formula (4.44) can also be derived from formula (4.42)

$$(\bar{I}\bar{a})_{\overline{n}|} = \lim_{m\to\infty} (I^{(m)}a)^{(m)}_{\overline{n}|} = \lim_{m\to\infty} \frac{\ddot{a}^{(m)}_{\overline{n}|} - n v^n}{i^{(m)}} = \frac{\bar{a}_{\overline{n}|} - n v^n}{\delta}.$$

In general, if the amount of the payment being made at exact moment t is $f(t)dt$, then an expression for the present value of an n-period continuous varying annuity would be

$$\int_0^n f(t)\, v^t\, dt. \tag{4.45}$$

An even more general continuous varying annuity would not only be one in which the payments are being made continuously and the variations are occurring continuously, but also one in which the force of interest is varying continuously. In this case, a generalized version of formula (4.45) would be

$$\int_0^n f(t)\, e^{-\int_0^t \delta_r dr}\, dt. \tag{4.46}$$

Example 4.18 Find an expression for the present value of a continuously increasing annuity with a term of n years if the force of interest is δ and if the rate of payment at time t is t^2 per annum.

The answer is obtained by performing an integration by parts

$$\begin{aligned}
\int_0^n t^2 e^{-\delta t} dt &= -\frac{t^2}{\delta} e^{-\delta t} \Big]_0^n + \frac{2}{\delta} \int_0^n t e^{-\delta t} dt \\
&= -\frac{n^2}{\delta} e^{-\delta n} - \frac{2t}{\delta^2} e^{-\delta t} \Big]_0^n + \frac{2}{\delta^2} \int_0^n e^{-\delta t} dt \\
&= -\frac{n^2}{\delta} e^{-\delta n} - \frac{2n}{\delta^2} e^{-\delta n} - \frac{2}{\delta^3} e^{-\delta t} \Big]_0^n \\
&= -\frac{n^2}{\delta} e^{-\delta n} - \frac{2n}{\delta^2} e^{-\delta n} - \frac{2}{\delta^3} e^{-\delta n} + \frac{2}{\delta^3} \\
&= \frac{2}{\delta^3} - e^{-\delta n} \left(\frac{n^2}{\delta} + \frac{2n}{\delta^2} + \frac{2}{\delta^3} \right).
\end{aligned}$$

4.9 SUMMARY OF RESULTS

Table 4.1 summarizes the expressions for present values of level annuities for various payment periods and interest conversion periods. The annuity illustrated pays 60 per annum for 10 years at 12% per annum.

Table 4.1 Summary of Relationships for Level Annuities in Chapter 4

Interest conversion period	*Payment period*			
	Annual	*Quarterly*	*Monthly*	*Continuous*
Annual	$60a_{\overline{10}\rceil .12}$	$60a^{(4)}_{\overline{10}\rceil .12}$	$60a^{(12)}_{\overline{10}\rceil .12}$	$60\bar{a}_{\overline{10}\rceil .12}$
Quarterly	$60\dfrac{a_{\overline{40}\rceil .03}}{s_{\overline{4}\rceil .03}}$	$15a_{\overline{40}\rceil .03}$	$15a^{(3)}_{\overline{40}\rceil .03}$	$15\bar{a}_{\overline{40}\rceil .03}$
Monthly	$60\dfrac{a_{\overline{120}\rceil .01}}{s_{\overline{12}\rceil .01}}$	$15\dfrac{a_{\overline{120}\rceil .01}}{s_{\overline{3}\rceil .01}}$	$5a_{\overline{120}\rceil .01}$	$5\bar{a}_{\overline{120}\rceil .01}$
Continuous	$60\dfrac{1 - e^{-1.2}}{e^{.12} - 1}$	$15\dfrac{1 - e^{-1.2}}{e^{.03} - 1}$	$5\dfrac{1 - e^{-1.2}}{e^{.01} - 1}$	$60\dfrac{1 - e^{-1.2}}{.12}$

EXERCISES

4.2 Annuities payable at a different frequency than interest is convertible

1. Find the accumulated value 18 years after the first payment is made of an annuity on which there are 8 payments of \$2000 each made at two-year intervals. The nominal rate of interest convertible semiannually is 7%. Answer to the nearest dollar.

2. Find the present value of a ten-year annuity which pays \$400 at the beginning of each quarter for the first 5 years, increasing to \$600 per quarter thereafter. The annual effective rate of interest is 12%. Answer to the nearest dollar.

3. A sum of \$100 is placed into a fund at the beginning of every other year for eight years. If the fund balance at the end of eight years is \$520, find the rate of simple interest earned by the fund.

4.3 Further analysis of annuities payable less frequently than interest is convertible

4. Rework Exercise 1 using the approach developed in Section 4.3.

5. Give an expression in terms of functions assuming a rate of interest per month for the present value, 3 years before the first payment is made, of an annuity on which there are payments of \$200 every 4 months for 12 years:
   a) Expressed as an annuity-immediate.
   b) Expressed as an annuity-due.

6. Show that the present value at time 0 of 1 payable at times 7, 11, 15, 19, 23, and 27 is
$$\frac{a_{\overline{28}|} - a_{\overline{4}|}}{s_{\overline{3}|} + a_{\overline{1}|}}.$$

7. A perpetuity of \$750 payable at the end of every year and a perpetuity of \$750 payable at the end of every 20 years are to be replaced by an annuity of R payable at the end of every year for 30 years. If $i^{(2)} = .04$, show that
$$R = 37{,}500 \left[\frac{1}{s_{\overline{2}|}} + \frac{v^{40}}{a_{\overline{40}|}} \right] \frac{s_{\overline{2}|}}{a_{\overline{60}|}}$$
   where all functions are evaluated at 2% interest.

8. Find an expression for the present value of an annuity-due of \$600 per annum payable semiannually for 10 years if $d^{(12)} = .09$.

9. The present value of a perpetuity paying 1 at the end of every three years is 125/91. Find i.

10. Find an expression for the present value of an annuity on which payments are \$100 per quarter for five years, just before the first payment is made, if $\delta = .08$.

11. A perpetuity paying 1 at the beginning of each year has a present value of 20. If this perpetuity is exchanged for another perpetuity paying R at the beginning of every two years, find R so that the values of the two perpetuities are equal.

12. Find an expression for the present value of an annuity on which payments are 1 at the beginning of each 4-month period for 12 years, assuming a rate of interest per 3-month period.

4.4 Further analysis of annuities payable more frequently than interest is convertible

13. Rework Exercise 2 using the approach developed in Section 4.4.

14. Derive formula (4.11).

15. Derive the following formulas analogous to formulas (3.6) and (3.12):

    a) $\dfrac{1}{a_{\overline{n}|}^{(m)}} = \dfrac{1}{s_{\overline{n}|}^{(m)}} + i^{(m)}$.

    b) $\dfrac{1}{\ddot{a}_{\overline{n}|}^{(m)}} = \dfrac{1}{\ddot{s}_{\overline{n}|}^{(m)}} + d^{(m)}$.

16. Derive the following formulas analogous to formulas (3.13) and (3.14):

    a) $\dot{a}_{\overline{n}|}^{(m)} = a_{\overline{n}|}^{(m)}(1+i)^{1/m}$.

    b) $\ddot{s}_{\overline{n}|}^{(m)} = s_{\overline{n}|}^{(m)}(1+i)^{1/m}$.

17. Derive the following formulas analogous to formulas (3.15) and (3.16):

    a) $\dot{a}_{\overline{n}|}^{(m)} = 1/m + a_{\overline{n-1/m}|}^{(m)}$.

    b) $\ddot{s}_{\overline{n}|}^{(m)} = s_{\overline{n+1/m}|}^{(m)} - 1/m$.

18. Express $\ddot{a}_{\overline{n}|}^{(12)}$ in terms of $a_{\overline{n}|}^{(2)}$ with an adjustment factor.

19. *a)* Show that $a_{\overline{n}|}^{(m)} = \dfrac{1}{m}\sum_{t=1}^{m} v^{t/m}\ddot{a}_{\overline{n}|}$.

    b) Verbally interpret the result obtained in (*a*).

20. A sum of \$10,000 is used to buy a deferred perpetuity-due paying \$500 every six months forever. Find an expression for the deferred period expressed as a function of d.

21. If $3a^{(2)}_{\overline{n}|} = 2a^{(2)}_{\overline{2n}|} = 45s^{(2)}_{\overline{1}|}$, find i.

22. Find an expression for the present value of an annuity which pays 1 at the beginning of each 3-month period for 12 years, assuming a rate of interest per 4-month period.

4.5 Continuous annuities

23. Find the value of t, $0 < t < 1$, such that 1 paid at time t is equivalent to 1 paid continuously between time 0 and 1.

24. Show algebraically and verbally that $a_{\overline{n}|} < a^{(m)}_{\overline{n}|} < \bar{a}_{\overline{n}|} < \ddot{a}^{(m)}_{\overline{n}|} < \ddot{a}_{\overline{n}|}$ where $m > 1$.

25. Find an expression for $\bar{a}_{\overline{n}|}$ if $\delta_t = \dfrac{1}{1+t}$.

26. There is \$40,000 in a fund which is accumulating at 4% per annum convertible continuously. If money is withdrawn continuously at the rate of \$2400 per annum, how long will the fund last?

27. If $\bar{a}_{\overline{n}|} = 4$ and $\bar{s}_{\overline{n}|} = 12$, find δ.

28. Show that $\dfrac{d}{dn} a_{\overline{n}|} = v^n / \bar{s}_{\overline{1}|}$.

4.6 Basic varying annuities

29. Verbally interpret the result obtained in Example 4.13.

30. Simplify $\sum_{t=1}^{20} (t+5)v^t$.

31. Show algebraically, and by means of a time diagram, the following relationship between $(Ia)_{\overline{n}|}$ and $(Da)_{\overline{n}|}$:

$$(Da)_{\overline{n}|} = (n+1)\,a_{\overline{n}|} - (Ia)_{\overline{n}|}.$$

32. The following payments are made under an annuity: 10 at the end of the fifth year, 9 at the end of the sixth year, decreasing by 1 each year until nothing is paid. Show that the present value is

$$\frac{10 - a_{\overline{14}|} + a_{\overline{4}|}\,(1 - 10i)}{i}.$$

33. Find the present value of a perpetuity under which a payment of 1 is made at the end of the first year, 2 at the end of the second year, increasing until a payment of n is made at the end of the nth year, and thereafter payments are level at n per year forever.

34. A perpetuity-immediate has annual payments of 1, 3, 5, 7. . . If the present value of the sixth and seventh payments are equal, find the present value of the perpetuity.

35. If X is the present value of a perpetuity of 1 per year with the first payment at the end of the second year and $20X$ is the present value of a series of annual payments $1, 2, 3, \ldots$ with the first payment at the end of the third year, find d.

36. An annuity-immediate has semiannual payments of $800, 750, 700, \ldots, 350$, at $i^{(2)} = .16$. If $a_{\overline{10}|\,.08} = A$, find the present value of the annuity in terms of A.

37. Annual deposits are made into a fund at the beginning of each year for 10 years. The first 5 deposits are \$1000 each and deposits increase by 5% per year therafter. If the fund earns 8% effective, find the accumulated value at the end of 10 years. Answer to the nearest dollar.

38. Find the present value of a 20-year annuity with annual payments which pays \$600 immediately and each subsequent payment is 5% greater than the preceding payment. The annual effective rate of interest is 10.25%. Answer to the nearest dollar.

4.7 More general varying annuities

39. Derive formula (4.42).

40. *a*) Find the sum of the payments in $(Ia)^{(12)}_{\overline{2}|}$.

    b) Find the sum of the payments in $(I^{(12)}a)^{(12)}_{\overline{2}|}$.

41. Show that $(I^{(m)}a)^{(m)}_{\overline{\infty}|} = \dfrac{1}{m(i^{(m)} - d^{(m)})}$.

42. Show that the present value of a perpetuity on which payments are 1 at the end of the 5th and 6th years, 2 at the end of the 7th and 8th years, 3 at the end of the 9th and 10th years, . . . , is
$$\frac{v^4}{i - vd}.$$

43. A perpetuity has payments at the end of each four-year period. The first payment at the end of four years is 1. Each subsequent payment is 5 more than the previous payment. It is known that $v^4 = 0.75$. Calculate the present value of this perpetuity.

44. A 10-year annuity has the following schedule of payments:

On each January 1	100
On each April 1	200
On each July 1	300
On each October 1	400

    Show that the present value of this annuity on January 1 just before the first payment is made is
$$1600\,\ddot{a}_{\overline{10}|}(I^{(4)}\ddot{a})^{(4)}_{\overline{1}|}.$$

45. A perpetuity provides payments every six months starting today. The first payment is 1 and each payment is 3% greater than the immediately preceding payment. Find the present value of the perpetuity if the effective rate of interest is 8% per annum.

4.8 Continuous varying annuities

46. Find the ratio of the total payments made under $(\bar{I}\bar{a})_{\overline{10}|}$ during the second half of the term of the annuity to those made during the first half.

47. Evaluate $(\bar{I}\bar{a})_{\overline{\infty}|}$ if $\delta = .08$.

48. Payments under a continuous perpetuity are made at the periodic rate of $(1 + k)^t$ at time t. The annual effective rate of interest is i, where $0 < k < i$. Find the present value of the perpetuity.

49. *a)* Find an integral expression for $(\bar{D}\bar{a})_{\overline{n}|}$.
    b) Find an expression not involving integrals for $(\bar{D}\bar{a})_{\overline{n}|}$.

50. A perpetuity is payable continuously at the annual rate of $1 + t^2$ at time t. If $\delta = .05$, find the present value of the perpetuity.

51. A one-year deferred continuous varying annuity is payable for 13 years. The rate of payment at time t is $t^2 - 1$ per annum, and the force of interest at time t is $(1 + t)^{-1}$. Find the present value of the annuity.

Miscellaneous problems

52. *a)* (1) Show that $\frac{d}{di} a_{\overline{n}|} = -v(Ia)_{\overline{n}|}$.
    (2) Find $\frac{d}{di} a_{\overline{n}|}$ evaluated at $i = 0$.
    b) (1) Show that $\frac{d}{di} \bar{a}_{\overline{n}|} = -v(\bar{I}\bar{a})_{\overline{n}|}$.
    (2) Find $\frac{d}{di} \bar{a}_{\overline{n}|}$ evaluated at $i = 0$.

53. *a)* Show that $\frac{i}{i^{(m)}} \fallingdotseq \frac{1}{1 - \frac{m-1}{2m} i} \fallingdotseq 1 + \frac{m-1}{2m} i$.

    b) Derive the following approximate equality:

$$a_{\overline{n}|}^{(m)} \fallingdotseq a_{\overline{n}|} + \frac{m-1}{2m}(1 - v^n).$$

54. For a given n, it is known that $\bar{a}_{\overline{n}|} = n - 4$ and $\delta = 10\%$. Find $\int_0^n \bar{a}_{\overline{t}|}\, dt$.

55. Find expressions for:

    a) $\sum_{t=1}^{n} (Ia)_{\overline{t}|}$.

    b) $\sum_{t=1}^{n} (Da)_{\overline{t}|}$.

56. A family wishes to provide an annuity of \$100 at the end of each month to their daughter now entering college. The annuity will be paid for only nine months each year for four years. Prove that the present value one month before the first payment is

$$1200\ddot{a}_{\overline{4}|}\ a^{(12)}_{\overline{9}|\overline{12}|}\ .$$

57. Show that

$$\sum_{r=1}^{h}\ (a_{\overline{rk}|}\ +s_{\overline{rk}|}\ )=\frac{1}{i}\left[\frac{s_{\overline{hk}|}}{a_{\overline{k}|}}-\frac{a_{\overline{hk}|}}{s_{\overline{k}|}}\right].$$

58. There are two perpetuities. The first has level payments of p at the end of each year. The second is increasing such that the payments are $q, 2q, 3q, \ldots$ Find the rate of interest which will make the difference in present value between these perpetuities:
    a) Zero.
    b) A maximum.

59. Fence posts set in soil last 9 years and cost \$2. Posts set in concrete last 15 years and cost \$2 + X. The posts will be needed for 35 years. Show that the break-even value of X, i.e. the value at which a buyer would be indifferent between the two types of posts is

$$2\left[\frac{a_{\overline{35}|}\ a_{\overline{15}|}}{a_{\overline{9}|}\ a_{\overline{45}|}}-1\right].$$

60. If $\bar{a}_{\overline{n}|}=a$ and $\bar{a}_{\overline{2n}|}=b$, express $(\bar{I}\bar{a})_{\overline{n}|}$ in terms of a and b.

5

Yield rates

5.1 INTRODUCTION

In Chapter 5 some important extensions of results covered in the earlier chapters are developed. These extensions involve concepts and techniques that are widely used in financial calculations in practice.

The technique of discounted cash flow analysis and the concept of yield rate are developed. The uniqueness of the yield rate is then considered. The use of these devices as tools for making financial decisions is stressed throughout.

Different techniques for measuring the interest return on an investment fund are developed and compared. The ramifications involved in the situation in which interest rates at the time of reinvestment differ from those prevailing at the time of initial investment are explored. Also developed is a generalization in which varying interest rates are a function of both the date of original investment and the time since investment. Finally, more general borrowing/lending models are considered for certain complex financial transactions.

The reader will find that this chapter presents a number of important results for using the theory of interest in more complex "real-world" contexts than considered in the earlier chapters. Also, Chapter 5 extends the applications of the theory of interest beyond borrowing and lending transactions to a broader range of business and financial transactions.

It is important to note that the effect of taxes is largely ignored throughout this book. The tax implications involved in business and financial transactions of the type considered herein vary considerably depending upon the nature of the transaction and the political jurisdictions involved. Nevertheless, tax considerations are quite important in practical applications of the theory of interest. It is often important to know whether a particular financial calculation

involving interest is determined on a "before-tax" or an "after-tax" basis. For example, if interest income is subject to taxation, "after-tax" rates will be significantly lower than "before-tax" rates. However, the principles contained in this book can be applied for calculations on either basis.

Another factor which may have a significant impact on financial calculations of the type discussed in this book is the treatment of expenses. Some rates of interest are quoted net of expenses and others are not. In the latter case, the net return to an investor will be reduced by the amount of expenses incurred. An example would be commissions and other fees involved in buying and selling securities.

5.2 DISCOUNTED CASH FLOW ANALYSIS

In Chapters 3 and 4 we analyzed the present value of certain types of annuities consisting of a regular series of payments. This approach can be generalized to any pattern of payments and is termed *discounted cash flow analysis*.

Consider a situation in which an investor makes deposits or contributions into an investment of $C_0, C_1, C_2, \ldots, C_n$ at times $0, 1, 2, \ldots, n$. For convenience, we assume that these times are equally spaced. If $C_t > 0$, then there is a net cash flow into the investment at time t; while if $C_t < 0$, there is a net cash flow out of the investment.

Sometimes it is more convenient to analyze a financial transaction in terms of withdrawals or returns from the investment rather than deposits or contributions into the investment. Thus, we can denote the returns as $R_0, R_1, R_2, \ldots, R_n$ at times $0, 1, 2, \ldots, n$. It is obvious that contributions and returns are equivalent concepts viewed from opposite sides of the transaction. Thus, we have

$$R_t = -C_t \text{ for } t = 0, 1, 2, \ldots, n. \tag{5.1}$$

Although it is not necessary to define C_t and R_t separately in this fashion, we will find it convenient to have both symbols available in developing certain formulas in this chapter.

It may happen that there is both a contribution and return at the same point in time. In this case, the two are offset against each other. For example, if we have a contribution of \$5000 at time 5 and also a return of \$1000 at time 5, then $C_5 = \$4000$ and $R_5 = -\$4000$.

We have chosen the time periods such that the investment begins at time $t = 0$ and ends at time $t = n$. Thus, if the investment is positive during this interval, we have $C_0 > 0$ ($R_0 < 0$) and $C_n < 0$ ($R_n > 0$). However, $C_t = -R_t$ may be either positive, negative, or zero for $t = 1, 2, \ldots, n - 1$.

To illustrate these definitions, consider a ten-year investment project in which the investor contributes \$10,000 at the beginning of the first year, \$5000 at the beginning of the second year, and then incurs maintenance expenses of \$1000 at the beginning of each remaining year thereafter. The project is expected to provide an investment return at the end of each year for the last five years of the project, starting at \$8000 and increasing \$1000 per year thereafter.

Table 5.1 summarizes the cash flows for this investment project. The last column contains values of R_t in order to display returns from the investment. However, the table could have been set up to contain values of C_t instead.

Table 5.1 Cash Flows for Investment Project Illustrated in Section 5.2

Year	*Contributions*	*Returns*	R_t
0	10,000	0	-10,000
1	5,000	0	- 5,000
2	1,000	0	- 1,000
3	1,000	0	- 1,000
4	1,000	0	- 1,000
5	1,000	0	- 1,000
6	1,000	8,000	7,000
7	1,000	9,000	8,000
8	1,000	10,000	9,000
9	1,000	11,000	10,000
10	0	12,000	12,000
Total	23,000	50,000	27,000

We now address the problem more generally. Assume that the rate of interest per period is i. Then the *net present value* at rate i of investment returns by the discounted cash flow technique is denoted by $P(i)$ and is given by

$$P(i) = \sum_{t=0}^{n} v^t R_t. \tag{5.2}$$

The value of $P(i)$ can be either positive or negative, depending on i. For the investment project illustrated in Table 5.1, $P(i)$ will be positive for "low" values of i and negative for "high" values of i. The positive net returns during the last few years of the investment will dominate the negative net returns during the early years when present values are computed at a "low" rate of interest, while the opposite is true at a "high" rate of interest.

A very important special case of formula (5.2) is the one in which $P(i) = 0$, i.e.

$$P(i) = \sum_{t=0}^{n} v^t R_t = 0. \tag{5.3}$$

The rate of interest i which satisfies formula (5.3) is called the *yield rate* on the investment. Stated in words:

The yield rate is that rate of interest at which the present value of returns from the investment is equal to the present value of contributions into the investment.

In the business and finance literature the yield rate is often called the *internal rate of return*. The terms "yield rate" and "internal rate of return" can be used interchangeably.

Yield rates are not an entirely new concept; we have encountered them before. The unknown rate of interest problems in Chapters 2, 3 and 4 can be characterized as yield rate problems. For example, we showed that the yield rate on an investment of \$16,000 which returns \$1000 at the end of every quarter for five years is 2.22623% per quarter, or a nominal rate of 8.9049% convertible quarterly in Example 3.8.

In this section so far we have adopted the vantage point of the investor, i.e. lender. However, if we are dealing with two-party transactions, then we could just as easily adopt the vantage point of the borrower. If this is done, then the values of C_t and R_t change signs.

However, the value of the yield rate given by formula (5.3) remains unchanged. Thus, the yield rate on a transaction is totally determined by the cash flows defined in that transaction and their timing, and is the same from either the borrower's or lender's perspective.

Yield rates are frequently used as an index to measure how favorable or unfavorable a particular transaction may be. From the lender's perspective, the higher the yield rate the more favorable the transaction. From the borrower's perspective, the opposite is the case. Although these convenient rules will usually produce reasonable results, Section 5.3, 5.8, and 5.9 contain illustrations in which difficulties arise in using yield rates in this fashion. Sections 5.8 and 5.9 contain a more systematic discussion of methods of comparing different financial transactions.

Yield rates need not be positive. If the yield rate is zero, then the investor (lender) received no return on investment. If the yield rate is negative, then the investor (lender) lost money on the investment. We will assume that such negative yield rates satisfy $-1 < i < 0$. It is difficult to find any practical interpretation for a situation in which $i < -1$, i.e. $1 + i < 0$.

Not all transactions are two-party transactions. For example, consider the investment project summarized in Table 5.1. Quite conceivably, the cash

outflows in this project were directed to multiple parties and/or the cash inflows arose from multiple sources. If that is the case, the yield rate calculation for the investor (lender) is still valid. However, there is not a single borrower on the other side of the transaction for which the same yield rate applies.

Another important consideration in using yield rates is to consider the period of time involved. For example, consider investing a sum of money under Options A and B. Option A credits 9% effective for five years, while Option B credits 8% effective for ten years. Which option should we choose as an investor?

To say that Option A is better than Option B because its yield rate is higher is naive. If we wish to invest for only five years, then the simple comparison of yield rates is valid. However, if we wish to invest for ten years, then we need to consider the rate at which we can reinvest the proceeds from Option A after the end of the first five years.

Thus, it is valid to use yield rates to compare alternative investments only if the period of investment is the same for all the alternatives. Section 5.4 contains a more comprehensive treatment of reinvestment rates.

The above definitions and formulas assume payments at integral periods of time. However, the results can easily be extended to include other regular or irregular intervals as well.

Solving for yield rates is analogous to solving for unknown rates of interest on annuities. In fact, if the payments constitute a basic annuity, then the techniques discussed in Section 3.8 can be applied directly.

If the payments do not constitute a basic annuity, the techniques would be similar. In general, an iteration method would typically be applied to the equation of value given by formula (5.3). The easiest way to solve such problems in practice would be to utilize a pocket calculator with built-in financial functions or to use a personal computer with a financial software package. Both of these options contain subroutines which readily handle such calculations.

Example 5.1 Find the yield rate for the investment project summarized in Table 5.1.

The equation of value is

$$1000(-10 - 5v - v^2 - v^3 - v^4 - v^5 + 7v^6 + 8v^7 + 9v^8 + 10v^9 + 12v^{10}) = 0.$$

The yield rate computed on a pocket calculator with built-in financial functions is found to be .1296, or 12.96%.

Example 5.2 Find the effective rate which the investment under Option A must earn for the second five years to be equivalent to the investment under Option B for the entire ten years for the illustration given in this section.

Let the rate in question be denoted by i. The equation of value is

$$(1.09)^5(1 + i)^5 = (1.08)^{10}$$

or

$$i = \frac{(1.08)^2}{1.09} - 1 = .0701, \text{ or } 7.01\%.$$

Thus, if the investor expects the prevailing interest rate to be greater than 7.01% at the end of 5 years, Option A would be the better choice. Otherwise, Option B would be preferred.

5.3 UNIQUENESS OF THE YIELD RATE

Our intuition leads us to expect that the yield rate as defined in Section 5.2 will be unique; and, in fact, in most commonly encountered financial transactions the yield rate is unique. However, transactions are occasionally encountered in which a yield rate is not unique.

As an example, consider a transaction in which a person makes payments of \$100 immediately and \$132 at the end of two years in exchange for a payment in return of \$230 at the end of one year. An equation of value for this transaction is

$$100(1 + i)^2 + 132 = 230(1 + i)$$

or

$$(1 + i)^2 - 2.3(1 + i) + 1.32 = 0.$$

Now factoring we obtain

$$[(1 + i) - 1.1][(1 + i) - 1.2] = 0.$$

Thus, the yield rate i is equal to either 10% or 20%!

The fact that transactions exist with multiple yield rates is difficult for most people to comprehend intuitively. However, it is not surprising when we recall formula (5.3)

$$P(i) = \sum_{t=0}^{n} v^t R_t = 0. \tag{5.3}$$

Formula (5.3) is an nth degree polynomial in v and could be written as an nth degree polynomial in i by simply multiplying both sides by $(1 + i)^n$. Of course, it is well-known that an nth degree polynomial has n roots (counting complex roots and roots of multiplicity m, where $m > 1$, as m roots). In the

example given immediately above, we have a quadratic with two distinct positive roots for i.

Since the yield rate is widely used as a measure of the financial value of a transaction, it is quite important in practice to be able to ascertain whether or not a yield rate is unique.

One very common situation in which the yield rate will be unique is when all cash flows in one direction are made before the cash flows in the other direction. Stated slightly more generally, this situation is one in which the net payments are all of one sign for the first portion of the transaction and then have the opposite sign for the remainder of the transaction.

Stated in mathematical terms, this situation can be characterized as one in which some value of k exists, $0 < k < n$, such that $R_t \leq 0$ for $t = 0, 1, 2, \ldots, k$ and $R_t \geq 0$ for $t = k+1, k+2, \ldots, n$. The financial transaction given in Table 5.1 is of this type with $n = 10$ and $k = 5$.

It can easily be shown that a yield rate in this situation will be unique. Looking at formula (5.3) as an nth degree polynomial, we see that there is only one sign change. From Descartes' rule of signs we know that there will be at most one positive real root. Since $v > 0$, then $i > -1$. Thus, the uniqueness will hold not only for positive values of i, but also for negative values of $i > -1$. This covers all values of concern, since values of $i < -1$ have no practical significance.

Descartes' rule of signs will also give us an upper bound on the number of multiple yield rates which may exist. The maximum number of yield rates is equal to the number of sign changes in the cash flows. Of course, the actual number of yield rates may well be less than the maximum.

Actually, yield rates are unique under a broader set of conditions than given above. It is possible to show that if the outstanding investment balance is positive at all points throughout the period of investment, then the yield rate will be unique.

Let B_t be the outstanding investment balance at time t where $t = 0, 1, 2, \ldots, n$. Then we have

$$B_0 = C_0 \tag{5.4}$$

and

$$B_t = B_{t-1}(1 + i) + C_t \text{ for } t = 1, 2, \ldots, n. \tag{5.5}$$

It is possible to show that if

1. $B_t > 0$ for $t = 0, 1, \ldots, n-1$, and
2. $i > -1$ exists such that formula (5.3) is satisfied,

then i is unique.

The proof of this result is as follows. The condition $i > -1$ is necessary to ensure that $1 + i$ is positive. Now rewrite formula (5.3) as

$$C_0(1 + i)^n + C_1(1 + i)^{n-1} + \cdots + C_{n-1}(1 + i) + C_n = 0.$$

We know that

$$\begin{array}{lcl}
B_0 = & & C_0 > 0 \\
B_1 = & B_0(1 + i) & + C_1 > 0 \\
B_2 = & B_1(1 + i) & + C_2 > 0 \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
B_{n-1} = & B_{n-2}(1 + i) & + C_{n-1} > 0 \\
B_n = & B_{n-1}(1 + i) & + C_n = 0.
\end{array}$$

By successive substitution in the above equations we have

$$B_n = C_0(1 + i)^n + C_1(1 + i)^{n-1} + \cdots + C_{n-1}(1 + i) + C_n = 0. \qquad (5.6)$$

This is the expected result, since the investment is exactly terminated at the end of n periods. Note that $C_0 > 0$ and $C_n < 0$, but that C_t for $t = 1, 2, \ldots, n - 1$ may be either positive, negative, or zero.

To prove the uniqueness of i, let $j > i$ be another yield rate. Let the outstanding investment at time t for interest rate j be denoted by B'_t. Then we have

$$\begin{array}{lcl}
B'_0 = & & C_0 = B_0 \\
B'_1 = & B'_0(1 + j) + C_1 & > B_0(1 + i) + C_1 = B_1 \\
B'_2 = & B'_1(1 + j) + C_2 & > B_1(1 + i) + C_2 = B_2 \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
B'_{n-1} = & B'_{n-2}(1 + j) + C_{n-1} & > B_{n-2}(1 + i) + C_{n-1} = B_{n-1} \\
B'_n = & B'_{n-1}(1 + j) + C_n & > B_{n-1}(1 + i) + C_n = B_n = 0.
\end{array}$$

But this is a contradiction, since B'_n must equal 0 if j is a yield rate. Thus j cannot be greater than i. The proof for $-1 < j < 1$ is analogous. This establishes the uniqueness of i.

Thus, if the outstanding investment balance is positive at all points throughout the period of investment, then the yield rate will be unique.

However, if the outstanding investment balance ever becomes negative at any one point, then a yield rate is not necessarily unique.

The situations in which multiple yield rates can occur may strike the reader as somewhat artificial and not very realistic of typical financial transactions. Although such situations are not common, they do occur in practice. A realistic example would be an investment in a physical plant which requires major renovation expenses midway through the period of investment. The resulting sign change in net cash flows may lead to multiple yield rates.

The discussion in this section has focused on the possibility of multiple yield rates. However, it is also possible that no yield rate exists or that all yield rates are imaginary. These possibilities are illustrated in Examples 5.3 and 5.4, respectively.

Readers who are interested in a more extensive discussion of the uniqueness of the yield rate are referred to papers by W. H. Jean (1968) and D. S. Promislow (1980) listed in the bibliography.

Example 5.3 A is able to borrow $1000 from B for one year at 8% effective and lend it to C for one year at 10% effective. What is A's yield rate on this transaction?

In this example, A is able to make a $20 profit at the end of one year in exchange for no net investment at all. Thus, no finite yield rate exists. We could say the yield rate is infinite. However, such a statement would not distinguish this transaction from an even more favorable one in which A is able to lend the $1000 to a fourth party D at 12% effective.

Example 5.4 What is the yield rate on a transaction in which a person makes payments of $100 immediately and $101 at the end of two years, in exchange for a payment of $200 at the end of one year?

An equation of value is

$$100(1 + i)^2 + 101 = 200(1 + i)$$

or

$$100i^2 = -1.$$

Thus, the yield rates are all imaginary numbers!

5.4 REINVESTMENT RATES

In Section 5.3 and the earlier chapters we have not directly considered the reinvestment by the lender of payments received from the borrower. This is equivalent to the implicit assumption that the lender can reinvest payments received from the borrower at a *reinvestment rate* equal to the original investment rate.

This may or may not be a valid assumption in practice depending upon the particular circumstances involved. If the lender is not able to reinvest the payments from the borrower at rates as high as the original investment, then the overall yield rate considering reinvestment will be lower than the stated yield rate. On the other hand, if the lender is able to reinvest such payments at even higher rates, then the overall yield rate will be higher than that stated.

Actually, we have already seen an example of a problem considering reinvestment rates. Example 5.2 considers the reinvestment of proceeds from Option A at the end of five years in order to make a valid comparison with Option B over the ten-year period in question. We now analyze two other situations in which reinvestment rates are directly taken into account.

First, consider the investment of 1 for n periods at rate i such that the interest is reinvested at rate j. It is desired to find the accumulated value at the end of n periods. This situation is illustrated in Figure 5.1.

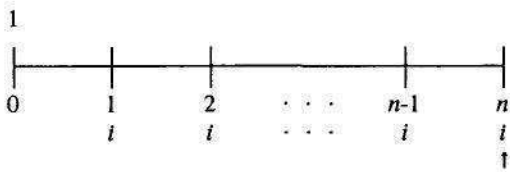

Figure 5.1 Time diagram for reinvestment rates involving an investment of 1 for n periods

The accumulated value at the end of n periods is equal to the principal plus the accumulated value of the interest, i.e.

$$1 + is_{\overline{n}|j}. \tag{5.7}$$

Formula (5.7) simplifies to the familiar $(1 + i)^n$ if $i = j$.

Second, consider the investment of 1 at the end of each period for n periods at rate i such that the interest is reinvested at rate j. It is desired to find the accumulated value of this annuity at the end of n periods. This situation is illustrated in Figure 5.2.

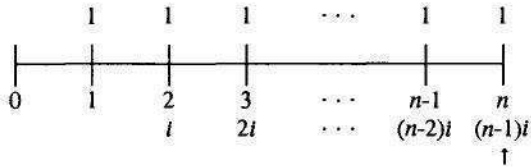

Figure 5.2 Time diagram for reinvestment rates involving an investment of 1 at the end of each period for n periods

The accumulated value of this annuity is equal to the sum of the annuity payments and the accumulated value of the interest, i.e.

$$n + i\,(Is)_{\overline{n-1}|j} = n + i\left[\frac{s_{\overline{n}|j} - n}{j}\right]. \tag{5.8}$$

Formula (5.8) simplifies to the familiar $s_{\overline{n}|}$ if $i = j$.

The consideration of reinvestment rates in financial calculations has become increasingly important and more widely used than heretofore. This reflects the greater volatility of interest rates in recent years, as well as the increased sophistication of investors.

An important consideration to a lender (investor) is the rate of repayment by the borrower. The "faster" the rate of repayment, the more significant the reinvestment issue becomes. The "slower" the rate of repayment, the longer the initial investment rate will dominate the calculation. This phenomenon is illustrated in Example 5.6. An analytical approach to the measurement of the rate of repayment is developed in Section 9.8.

One final observation is that the results of financial calculations involving reinvestment rates are dependent upon the period of time under consideration. Thus, it is important to specify the period of time for which calculations are being made when reinvestment rates are being taken into account.

Example 5.5 Payments of $1000 are invested at the beginning of each year for 10 years. The payments earn interest at 7% effective and the interest can be reinvested at 5% effective. (1) Find the amount in the fund at the end of 10 years. (2) Find the purchase price an investor should pay to produce a yield rate of 8% effective.

1. A variation of formula (5.8) appropriate for an annuity-due is

$$n + i(Is)_{\overline{n}|j} = n + i\left[\frac{s_{\overline{n+1}|j} - (n+1)}{j}\right]$$

Thus, the amount of the fund at the end of 10 years is

$$1000\left[10+.07\left(\frac{s_{\overline{11}|.05}-11}{.05}\right)\right]=1000\left[10+.07\left(\frac{14.2068-11}{.05}\right)\right]=\$14{,}490$$

to the nearest dollar. The answer lies between $1000\ddot{s}_{\overline{10}|.05}=\$13{,}207$ and $1000\ddot{s}_{\overline{10}|.07}=\$14{,}784$, as would be expected.

2. The purchase price to yield 8% effective would be

$$14{,}490(1.08)^{-10}=\$6712$$

to the nearest dollar.

Example 5.6 Compare the yield rates on the three loan repayment schedules described in Example 3.3, if the repayments to the lender can be reinvested at only 7% instead of 9% as earned on the original loan.

1. The accumulated value of all payments at the end of 10 years is

$$1000(1.09)^{10}=\$2367.36.$$

The yield rate i is found from the equation of value

$$1000(1+i)^{10}=\$2367.36$$

which is immediately seen to be $i=.09$. In this case, the reinvestment risk totally disappears, since the borrower makes no payments until the end of the period of the loan.

2. The accumulated value of all payments at the end of 10 years is

$$1000+90s_{\overline{10}|.07}=1000+90(13.8164)=\$2243.48$$

by a direct application of formula (5.7). The yield rate i is found from the equation of value

$$1000(1+i)^{10}=\$2243.48$$

which gives $i=.0842$, or 8.42%. This answer is less than the answer to case 1, since the reinvestment rate of 7% has an effect in this case.

3. The accumulated value of all payments at the end of 10 years is

$$\left(\frac{1000}{a_{\overline{10}|.09}}\right)s_{\overline{10}|.07}=(155.82)(13.8164)=\$2152.88.$$

The yield rate i is found from the equation of value

$$1000(1+i)^{10}=\$2152.88$$

which gives $i=.0797$, or 7.97%. This answer is less than the answer to case 2, since the repayment schedule for case 3 is "faster" than case 2 which increases the

effect of the reinvestment rate on the answer. Note that the yield rate still exceeds 7%, as would be expected.

5.5 INTEREST MEASUREMENT OF A FUND

A significant problem in practical work is the determination of the yield rate earned by an investment fund. Recall that the basic definition of an effective rate of interest given in Section 1.3 assumed that the principal remains constant throughout the period and that all the interest earned is paid at the end of the period. In practice these assumptions are often not satisfied. It is common for a fund to be incremented with new principal deposits, decremented with principal withdrawals, and incremented with interest earnings many times throughout a period, often at irregular intervals. Some method must be devised for these situations to determine reasonable effective rates of interest.

Consider finding the effective rate of interest earned by a fund over one measurement period. We make the following definitions:

A = the amount in the fund at the beginning of the period

B = the amount in the fund at the end of the period

I = the amount of interest earned during the period

C_t = the net amount of principal contributed at time t (positive or negative), where $0 \le t \le 1$

C = the total net amount of principal contributed during the period (positive or negative), i.e.

$$C = \sum_t C_t$$

${}_a i_b$ = the amount of interest earned by 1 invested at time b over the following period of length a, where $a \ge 0$, $b \ge 0$, and $a + b \le 1$

Note that in terms of the notation used in Section 5.3 $A = B_0$ and $B = B_1$. We use the alternative notation involving A and B in order to develop a formula in widespread use with its traditional notation.

The fund at the end of the period must equal the fund at the beginning of the period plus net principal contributed (positive or negative) plus interest earned, i.e.

$$B = A + C + I. \tag{5.9}$$

To be consistent with the definition of the effective rate of interest given in Section 1.3 we will assume that all the interest earned I is received at the end of the period. Then an exact equation of value for the interest earned over the period, $0 \le t \le 1$, is

$$I = iA + \sum_t C_t \cdot {}_{1-t}i_t. \tag{5.10}$$

Unfortunately, formula (5.10) is not in a form which can be directly solved for i. It is necessary to find values for ${}_{1-t}i_t$. Assuming compound interest throughout the period, we have

$${}_{1-t}i_t = (1+i)^{1-t} - 1. \tag{5.11}$$

We can substitute formula (5.11) into formula (5.10) obtaining an exact equation for i. This equation can be solved by iteration. Section 5.3 guarantees that the rate found by iteration will be unique as long as the fund balance never becomes negative.

If a computer or pocket calculator with built-in financial functions is not available to do the iteration, or if only approximate answers are required, it is possible to produce a simplified formula by making the assumption that

$${}_{1-t}i_t \fallingdotseq (1-t)i. \tag{5.12}$$

Formula (5.12) is a version of simple interest for this situation. We can substitute formula (5.12) into formula (5.10) and solve for i, obtaining

$$i \fallingdotseq \frac{I}{A + \sum_t C_t(1-t)}. \tag{5.13}$$

The numerator of formula (5.13) is the amount of interest earned on the fund. The denominator can be interpreted as the average amount of principal invested and is often called the *exposure associated with i*. Although formula (5.13) does not produce a true effective rate of interest because of the simple interest assumption, it will generally produce results quite close to a true effective rate of interest as long as the C_t's are small in relation to A, which is often the case in practice. However, if the C_t's are not small in relation to A, then the error can become significant.

Formula (5.13) is in a form which can be directly calculated. However, the summation term in the denominator is often rather laborious. Therefore, a further simplifying assumption is often made; namely, that principal deposits and withdrawals occur uniformly throughout the period. Thus, on average, we might assume that net principal contributions occur at time $t = 1/2$. If this assumption is made, then formula (5.13) becomes

$$\begin{aligned} i &\fallingdotseq \frac{I}{A + .5C} \\ &= \frac{I}{A + .5(B - A - I)} \quad \text{from formula (5.9)} \end{aligned}$$

$$= \frac{2I}{A + B - I}. \tag{5.14}$$

Figure (5.3) is an illustrative time diagram for this formula.

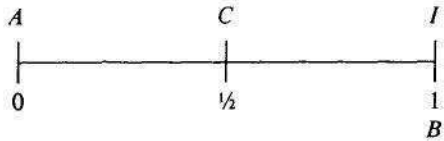

Figure 5.3 Time diagram for formula (5.14)

Formula (5.14) is an important formula which is widely used in practice to calculate earned rates of interest, e.g. it has been used by insurance regulators to compute the yield rate on invested assets of insurance companies. It is a very convenient formula, since it involves only A, B, and I, which are readily available. However, it should be remembered that it does assume that net principal contributions occur at time $t = 1/2$. If this assumption is not warranted, then the more exact (but still approximate) formula (5.13) should be used.

In some cases it is possible to develop simplified versions of formula (5.13) which will be more accurate than formula (5.14). For example, if it is known that net principal contributions occur at time k on the average, $0 < k < 1$, then a generalization of formula (5.14) is given by

$$i \fallingdotseq \frac{I}{kA + (1-k)B - (1-k)I}. \tag{5.15}$$

The derivation of formula (5.15) is left as an exercise. It is immediately clear that formula (5.15) becomes formula (5.14) when $k = 1/2$. However, if we know that net principal contributions occur on April 1 on the average, then for calendar year calculations the use of formula (5.15) with $k = 1/4$ should produce superior answers to formula (5.14).

Formula (5.12) looks very similar to simple interest as defined in Section 1.4. However, it can be shown that the two are not equivalent by considering the form of δ_t under each assumption.

As defined in Section 1.4, the accumulation function for simple interest is given by

$$a(t) = 1 + ti. \tag{1.5}$$

This is equivalent to the assumption that

$${}_t i_0 = ti. \tag{5.16}$$

An expression for δ_t under this assumption is given by formula (1.34)

$$\delta_t = \frac{i}{1 + ti}. \tag{1.34}$$

For the version of simple interest defined by formula (5.12) we have

$$e^{\int_t^1 \delta_r dr} = 1 + {}_{1-t}i_t = 1 + (1 - t)i$$

or

$$\int_t^1 \delta_r dr = \log_e[1 + (1 - t)i]$$

and differentiating with respect to t

$$\delta_t = \frac{i}{1 + (1 - t)i} \quad \text{for } 0 \le t \le 1. \tag{5.17}$$

Clearly formulas (1.34) and (5.17) are not equivalent; in fact, they are equal only for $t = 1/2$. In addition, it should be noted that formula (1.34) is a decreasing function of t, while formula (5.17) is an increasing function of t.

It is possible to develop analogous results for funds in which payments are being made continuously. Let B_t be the outstanding fund balance at time t, $0 \le t \le n$, and assume that contributions (positive or negative) are being made continuously at exact time t at the rate of C_t per period. Then a generalized version of formula (5.6) is given by

$$B_n = B_0(1 + i)^n + \int_0^n C_t(1 + i)^{n-t} dt. \tag{5.18}$$

In essence, formula (5.18) says that the fund balance at the end of n measurement periods is equal to the beginning fund balance accumulated with interest for n periods, plus the accumulated value of all the intervening payments (positive or negative) in the amount of $C_t\,dt$ accumulated with interest to the end of the n periods.

An even more general formula can be obtained by also allowing the force of interest to vary continuously. A generalized version of formula (5.18) would then be given by

$$B_n = B_0 e^{\int_0^n \delta_s ds} + \int_0^n C_t e^{\int_t^n \delta_s ds} dt. \tag{5.19}$$

The following differential equation is associated with formula (5.19)

$$\frac{d}{dt}B_t = \delta_t B_t + C_t. \tag{5.20}$$

Formula (5.20) has an interesting verbal interpretation. The left-hand side is the instantaneous rate of change in the fund balance at time t. The right-hand side attributes this instantaneous rate of change to two factors: (1) interest at

force δ_t on the fund balance B_t, plus (2) the rate of contribution (positive or negative) to the fund at exact time t. The derivation of formula (5.20) is left as an exercise.

Formulas such as (5.18), (5.19), and (5.20) are instructive conceptually. However, they are not widely used in practical applications.

Example 5.7 ***At the beginning of the year an investment fund was established with an initial deposit of $1000. A new deposit of $500 was made at the end of four months. Withdrawals of $200 and $100 were made at the end of six months and eight months, respectively. The amount in the fund at the end of the year is $1272. Find the approximate effective rate of interest earned by the fund during the year, using formula (5.13).***

The interest earned I is computed to be

$$1272 - (1000 + 500 - 200 - 100) = 72.$$

Applying formula (5.13) we have

$$i = \frac{72}{1000 + \frac{2}{3}\cdot 500 - \frac{1}{2}\cdot 200 - \frac{1}{3}\cdot 100} = \frac{72}{1200} = .06, \text{ or } 6\%.$$

Example 5.8 ***Find the effective rate of interest earned during a calendar year by an insurance company with the following data:***

Assets, beginning of year	***$10,000,000***
Premium income	***1,000,000***
Gross investment income	***530,000***
Policy benefits	***420,000***
Investment expenses	***20,000***
Other expenses	***180,000***

It is conventional practice to offset investment expenses against gross investment income. Thus, we have the following:

$$A = 10{,}000{,}000$$
$$B = 10{,}000{,}000 + 1{,}000{,}000 + 530{,}000 - 420{,}000 - 20{,}000 - 180{,}000 = 10{,}910{,}000$$
$$I = 530{,}000 - 20{,}000 = 510{,}000.$$

Therefore, using formula (5.14)

$$i = \frac{2(510{,}000)}{10{,}000{,}000 + 10{,}910{,}000 - 510{,}000} = .05, \text{ or } 5\%.$$

5.6 TIME-WEIGHTED RATES OF INTEREST

The methods for computing the yield rate earned by an investment fund outlined in Section 5.5 are sensitive to the amounts of money invested during various subperiods when the investment experience is volatile during the year. For example, if "large" amounts happen to be invested when the earnings on the fund are "high" and "small" amounts when the earnings are "low," the overall yield rate will be quite favorable. The reverse situation will, of course, produce the opposite result.

We will demonstrate this phenomenon with an extreme illustration. Assume an investor has an investment fund in which a \$1000 investment is worth only \$500 at the end of six months, but is worth \$1000 again at the end of the year. If no principal is deposited or withdrawn during the year, then the yield rate for the entire year is obviously zero.

Now consider what happens if the investor doubles the outstanding investment at the end of six months. The original \$1000 is worth only \$500 at the end of six months, so the investor deposits another \$500 at that time. The new balance of \$1000 is then worth \$2000 at the end of the year. The equation of value for this transaction is

$$1000(1 + i) + 500(1 + i)^{1/2} = 2000.$$

This equation can be solved as a quadratic in $(1 + i)^{1/2}$, which produces the yield rate $i = .4069$, or 40.69%.

Next consider what happens if the investor halves the outstanding investment at the end of six months. The original \$1000 is worth only \$500 at the end of six months, so the investor withdraws \$250 at that time. The new balance of \$250 is then worth \$500 at the end of the year. The equation of value for this transaction is

$$1000(1 + i) - 250(1 + i)^{1/2} = 500.$$

Again, solving the quadratic in $(1 + i)^{1/2}$ produces the yield rate $i = -.2892$, or -28.92%.

It would seem that the underlying yield rate for the fund based on its actual investment performance should be zero. However, in the first illustration above the computed yield rate is considerably greater than zero because the investor deposited principal just as the investment experience was about to become very favorable. In the second illustration just the opposite happened. The investor withdrew principal which caused the yield rate to become significantly negative.

Since the amount invested clearly affects the computed yield rate, rates computed by the methods in Section 5.5 are sometimes called *dollar-weighted*

rates of interest. It is important to observe that compound interest calculations as developed in previous chapters are done on this basis.

Now assume that the investment decisions for the fund are being made by an investment manager, while the decisions to deposit or withdraw principal are made by the owner of the fund. Although the dollar-weighted calculations in the above two illustrations provide an accurate measure of the actual return realized by the owner of the fund, they do not provide a good measure of the "true" performance of the investment manager, which was zero.

Such a measure is provided by an alternative basis for calculating fund yields called *time-weighted rates of interest*. In this method we consider successive subintervals of the year each time a deposit or withdrawal is made. Thus, in the illustrations given above, the yield rate for the first six months of the year is $j_1 = -50\%$ and for the second six months is $j_2 = 100\%$. We can combine these for the entire year to obtain

$$1 + i = (1 + j_1)(1 + j_2) = (1 - .5)(1 + 1) = 1.$$

Thus, $i = 0$ regardless of when principal is deposited or withdrawn.

We can generalize this approach as follows. Assume that $m - 1$ principal deposits or withdrawals are made during the year at times $t_1, t_2, \ldots, t_{m-1}$. This will divide the year into m subintervals as illustrated in Figure 5.4.

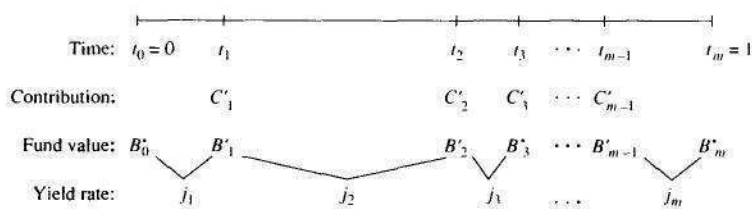

Figure 5.4 Time diagram for time-weighted rates of interest

Let the amount of the net contribution to the fund (positive or negative) at time t_k be denoted by C'_k for $k = 1, 2, \ldots, m - 1$. The prime on the symbol is being used to distinguish this notation from that used earlier in the chapter, i.e. C'_k is actually equal to C_{t_k}.

Let the fund values immediately before each contribution to the fund be denoted by B'_k for $k = 1, 2, \ldots, m - 1$. Also the fund value at the beginning of the year is denoted by $B'_0 = B_0$, while the fund value at the end of year is

denoted by $B'_m = B_1$. Finally, let the yield rates over the m subintervals be denoted by j_k for $k = 1, 2, \ldots, m$.

The yield rates over the m subintervals by the time-weighted method are given by

$$1 + j_k = \frac{B'_k}{B'_{k-1} + C'_{k-1}}$$

or

$$j_k = \frac{B'_k}{B'_{k-1} + C'_{k-1}} - 1 \text{ for } k = 1, 2, \ldots, m. \tag{5.21}$$

In words, 1 plus the yield rate for each subinterval is equal to the fund balance at the end of the subinterval divided by the fund balance at the beginning of the subinterval.

The overall yield rate for the entire year is then given by

$$1 + i = (1 + j_1)(1 + j_2) \cdots (1 + j_m)$$

or

$$i = (1 + j_1)(1 + j_2) \cdots (1 + j_m) - 1. \tag{5.22}$$

It is important to note that yield rates computed by the time-weighted method are not consistent with an assumption of compound interest. Nevertheless, time-weighted calculations do provide better indicators of underlying investment performance than dollar-weighted calculations. However, dollar-weighted calculations provide a valid measure of the actual investment results achieved.

Example 5.9 On January 1 an investment account is worth $100,000. On May 1 the value has increased to $112,000 and $30,000 of new principal is deposited. On November 1 the value has declined to $125,000 and $42,000 is withdrawn. On January 1 of the following year the investment account is again worth $100,000. Compute the yield rate by: (1) the dollar-weighted method, and (2) the time-weighted method.

Figure 5.5 illustrates the transactions involved in this example.

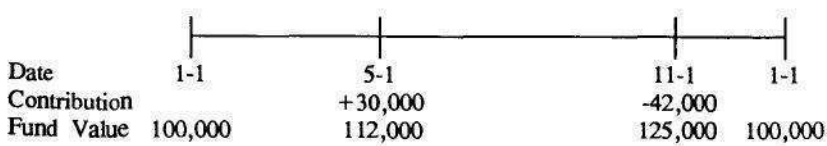

Figure 5.5 Time diagram for Example 5.9

1. The total amount of interest earned can be found from formula (5.9)

$$B = A + C + I$$

which gives

$$100{,}000 = 100{,}000 + (30{,}000 - 42{,}000) + I$$

or

$$I = 12{,}000.$$

We will use formula (5.13) to find the dollar-weighted rate of interest

$$i \doteq \frac{12{,}000}{100{,}000 + \frac{2}{3} \cdot 30{,}000 - \frac{1}{6} \cdot 42{,}000}$$
$$= \frac{12{,}000}{113{,}000} = .1062, \text{ or } 10.62\%.$$

A more refined answer could be obtained by using compound interest instead of formula (5.13), but solving the equation of value would require a difficult iteration. The extra effort is not worth the trouble for purposes of this example. It is important to note that the intermediate fund balances ($112,000 and $125,000 in this example) do not have any effect on the answer when using the dollar-weighted method.

2. Using formulas (5.21) and (5.22) the time-weighted rate of interest is found to be

$$j = \left[\frac{112{,}000}{100{,}000}\right]\left[\frac{125{,}000}{142{,}000}\right]\left[\frac{100{,}000}{83{,}000}\right] - 1$$
$$= (1.12)\,(.880282)\,(1.204819) - 1$$
$$= .1879, \text{ or } 18.79\%.$$

Thus, the time-weighted rate of interest is dramatically higher than the dollar-weighted rate of interest. The reason for this becomes evident upon analyzing the three subintervals. Investment experience was very favorable during the first four months and the last two months of the year. However, it was quite adverse during the intervening six months. Since new principal was deposited just before the experience was about to turn sour and principal was withdrawn just before the experience was about to become favorable again, the dollar-weighted calculation was quite adversely affected.

In summary, the dollar-weighted yield, 10.62%, is a measure of the actual financial results achieved by the investor. The time-weighted yield, 18.79%, is a measure of the actual performance of the investment fund independent of the amount that happens to be invested.

The reader should not think that differences of this magnitude between the two methods are typical. It took the combination of bad timing in making principal

deposits and withdrawals, together with highly volatile investment performance (such as could occur in a fund heavily invested in common stocks), to create such a dramatic effect. The differences between the two methods would be much smaller in the event that either the investment performance was more stable, or that principal deposits and withdrawals were smaller in relation to the fund balance, or both. In fact, in stable investment funds invested at stable interest rates, the difference between the two methods would generally be insignificant.

5.7 PORTFOLIO METHODS AND INVESTMENT YEAR METHODS

Consider the commonly-encountered situation in which an investment fund is being maintained for a number of different entities, i.e. individuals or companies. An example would be a pension fund in which each plan participant has an individual account. However, the investment fund is commingled, i.e. each account does not have its own separate group of segregated assets, but rather a pro rata share of the entire fund.

An issue arises in connection with the crediting of interest to the various accounts. Two distinctly different approaches to allocating interest to the various accounts are in common use; namely, the *portfolio method* and the *investment year method.*

Under the portfolio method an average rate based on the earnings of the entire fund is computed and credited to each account. This method is quite straightforward and simple to implement. It is a method of long-standing usage in a variety of different situations.

However, problems arise in using the portfolio method during periods of fluctuating interest rates. For example, consider a situation in which interest rates have risen significantly in the recent past. The portfolio method might produce an average rate of 8%, while new deposits might be able to earn 10% on their own. The portfolio rate is lower because the fund includes a collection of lower yielding investments made in the past. In this situation there is a significant disincentive for anyone to make new deposits to the fund, and there is also an increased incentive for withdrawals.

The investment year method was developed to handle this problem by recognizing the date of investment, as well as the current date, in crediting interest. It is a newer method that came into vogue during the 1960s and 1970s when there was a long period of rising interest rates. The rate on new deposits under the investment year method (10% in the above example) is often called the *new money rate.*

The investment year method is obviously more complicated than the portfolio method to apply in practice. However, many financial institutions, such as banks and insurance companies, felt it necessary to utilize the

investment year method to attract new deposits and discourage withdrawals during periods of rising interest rates. Of course, when interest rates decline the situation will reverse and the portfolio method will be more attractive than the investment year method. When interest rates fluctuate a lot up and down, as they did during the 1980s, it becomes an interesting guessing game as to which method will produce the more favorable results.

In applying the investment year method an immediate problem arises in connection with reinvestment rates. Two general approaches to this problem have been developed in practice. Under the *declining index system*, the funds associated with a particular investment year decline as the need to reinvest the money occurs. The interest rate credited under the investment year method reflects the investment rate on the remaining assets which are dwindling.

By contrast, under the *fixed index system* the funds associated with a particular investment year remain fixed in amount. The interest rate credited under the investment year method reflects the investment rate on the original investment modified by subsequent reinvestment rates.

Another consideration in implementing the investment year method is the need to truncate the process at some point. To illustrate in the extreme, it makes little sense to attempt to maintain an investment year method for 100 years! Normally, an arbitrary period is chosen after which time the process stops and reverts to the portfolio method. For example, if the period for which the investment year method is applicable is chosen to be 10 years, then any funds on deposit more than 10 years will be credited on a portfolio basis.

In practice, what is normally done to implement the investment year method is to specify a two-dimensional table of interest rates by date of original investment and time elapsed since that date. In order to simplify the presentation, we will assume that these periods are measured in calendar years and that all deposits and withdrawals are made on January 1.

Let y be the calendar year of deposit and let m be the number of years for which the investment year method is applicable. The rate of interest credited for the tth year of investment is denoted by i_t^y for $t = 1, 2, \ldots, m$. For $t > m$ the portfolio method is applicable and interest rates vary by calendar year only. The portfolio rate of interest credited for calendar year y is denoted by i^y. This notation is a generalization of the notation developed in formula (1.4*b*) and used subsequently in Chapter 1.

Table 5.2 is an illustrative array of rates credited under the investment year method with $m = 5$. The first year in the table is calendar year z and the most recent year is calendar year $z + 10$.

Table 5.2 (29p6 x 20p9)

Calendar year of original investment y	*Investment year rates*					*Portfolio rates*	*Calendar year of portfolio rate*
	i_1^y	i_2^y	i_3^y	i_4^y	i_5^y	i^{y+5}	$y+5$
z	8.00%	8.10%	8.10%	8.25%	8.30%	8.10%	$z+5$
$z+1$	8.25	8.25	8.40	8.50	8.50	8.35	$z+6$
$z+2$	8.50	8.70	8.75	8.90	9.00	8.60	$z+7$
$z+3$	9.00	9.00	9.10	9.10	9.20	8.85	$z+8$
$z+4$	9.00	9.10	9.20	9.30	9.40	9.10	$z+9$
$z+5$	9.25	9.35	9.50	9.55	9.60	9.35	$z+10$
$z+6$	9.50	9.50	9.60	9.70	9.70		
$z+7$	10.00	10.00	9.90	9.80			
$z+8$	10.00	9.80	9.70				
$z+9$	9.50	9.50					
$z+10$	9.00						

The pattern of interest rates for a particular year of investment follows a horizontal line to the right-hand column and then downward. For example, in Table 5.2 the solid lines are the successive interest rates credited for deposits made at the beginning of calendar year $z + 2$.

The interest rates credited in any particular calendar year appear on an upward diagonal to the right. For example, in Table 5.2 the dashed lines are the various rates of interest credited in calendar year $z + 7$.

Competition among investment funds generally is focused on new money rates to be credited in the first year in order to entice depositors to invest in that particular fund. The new money rates appear in the column headed i_1^y and are connected by dotted lines in Table 5.2 for calendar years z through $z + 10$ inclusive.

In practice, implementing an investment year method is generally more complex than may be implied by Table 5.2. One source of complexity is that investment funds often change their credited rates more frequently than annually, e.g. monthly or quarterly. Another complication is the need to handle deposits or withdrawals at any date. Typically, the credited rates are for calendar periods and the interest credited would be based on the periods funds are invested at the various rates.

It is also worth noting that the investment year method illustrated in Table 5.2 is probably based on the fixed index system. If the declining index system were being utilized, the rates going horizontally through the five-year period for each calendar year of original investment would be more nearly constant than in Table 5.2.

***Example 5.10** An investment of \$1000 is made at the beginning of calendar year $z + 4$ in an investment fund crediting interest according to the rates contained in Table 5.2. How much interest is credited in calendar years $z + 7$ through $z + 9$ inclusive?*

We can readily adapt the approach taken in formula (1.38) to compute accumulated values at varying rates of interest to this situation. The accumulated value of the investment at the beginning of calendar year $z + 7$ is

$$1000(1.09)(1.091)(1.092) = \$1298.60.$$

The accumulated value of the investment at the beginning of calendar year $z + 10$ is

$$1000(1.09)(1.091)(1.092)(1.093)(1.094)(1.091) = \$1694.09.$$

Thus, the total amount of interest credited in calendar years $z + 7$ through $z + 9$ is

$$1694.09 - 1298.60 = \$395.49.$$

5.8 CAPITAL BUDGETING

A problem facing both individual and corporate investors is the need to determine the amount of capital to invest and the allocation of that capital among various alternative investments. The process of making such financial decisions is often called *capital budgeting*.

In practice, two major approaches to capital budgeting are most commonly encountered. The first of these is the *yield rate method.* In this method the investor computes the yield rate for each alternative investment using formula (5.3). The investor establishes an *interest preference rate*, which is the minimum acceptable rate of return. Setting the interest preference rate is a matter of business judgment involving considerations such as the cost of raising capital and the investor's profit objectives.

Investments with yield rates higher than the interest preference rate are considered further, while investments with yield rates that are lower are rejected. The various alternative investments with rates higher than the interest preference rate are ranked and those with the highest yield rates are selected in descending order until the amount of capital available for investment is exhausted.

The second approach is the *net present value method*. In this method the investor computes $P(i)$ for each alternative investment using formula (5.2). $P(i)$ is calculated at the interest preference rate as described above.

Investments with a positive $P(i)$ are considered further, while investments with a negative $P(i)$ are rejected. Capital is then allocated among those investments with a positive $P(i)$ in such a manner that the total present value of returns from the investment minus contributions to the investment is maximized. The present values are computed at the interest preference rate.

If a unique yield rate exists, then these two approaches will produce consistent results. In other words, investments with yield rates higher than the interest preference rate will have a positive $P(i)$ and conversely. However, the fact that the yield rate may not always exist and be unique has led many writers in finance to favor the net present value method over the yield rate method.

Another argument that has been advanced to favor the net present value method is that it automatically maximizes dollar returns to the investor as part of the decision process. On the other hand, the yield rate method has the appeal of using numbers that are very easy to grasp and compare. However, the use of the yield rate method does not lead directly to financial results measured in terms of dollars without making additional calculations.

The above description of capital budgeting has been viewed from the perspective of an investor (lender), which is typically the manner in which capital budgeting is applied. However, it is possible to adapt the procedure for use by a borrower. In this case, the rules for the yield rate method work in the opposite direction, i.e. a "favorable" transaction has a low yield rate while an "unfavorable" transaction has a high yield rate. On the other hand, the net present value method can be applied in the same fashion for borrowers as for lenders as long as the R_t's in formula (5.2) are from the borrower's perspective.

The discussion of capital budgeting in this section has not considered the comparative risk involved in the various alternative investments. In essence, we are assuming that risk is non-existent in the alternative investments being compared. Consideration of risk may well modify the decision process. For example, it is doubtful that many investors would (or should) prefer a high-risk investment with a projected yield rate of 15% to low-risk investment with a projected yield rate of 14%. The subject of financial calculations involving risk is considered further in Section 9.5.

The description of capital budgeting given in this section is quite brief. Readers interested in a more extensive treatment are referred to any of several standard textbooks in finance.

Example 5.11 Analyze the investment project given in Table 5.1 as a capital budgeting exercise.

We know from Example 5.1 that the yield rate on this project is 12.96%. Table 5.3 tabulates the net present value, $P(i)$, at a wide range of illustrative interest rates using formula (5.2).

Table 5.3 Net Present Values for Example 5.11

Rate of interest i	*Net present value P(i)*
0%	$ 27,000
5	12,675
10	3,695
15	- 2,046
20	- 5,778
25	- 8,236

Let us assume that an investor has an interest preference rate equal to 10%. Using the yield rate method the investor would accept this project for further consideration, since $12.96\% > 10\%$. Using the net present value method the investor also would accept it, since $P(.1) = 3695 > 0$.

Now, consider an investor with an interest preference rate equal to 15%. Using the yield rate method the investor would reject this project since $12.96\% < 15\%$. Using the net present value method the investor also would reject, since $P(.15) = -2046 < 0$.

It is instructive to analyze Table 5.3 graphically. These results are displayed as the solid line in Figure 5.6.

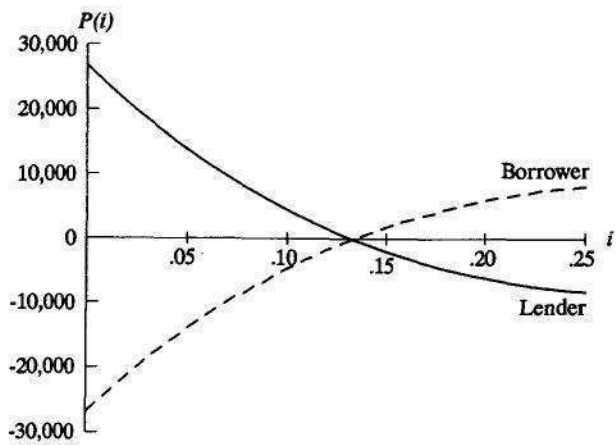

Figure 5.6 Net present values for Example 5.11

It is clear that $P(i)$ is a decreasing function of the rate of interest. Moreover, it is positive to the left of the yield rate 12.96%, and is negative to the right. The solid line in Figure 5.6 is a typical graph for the value of a financial transaction with a unique yield rate to the investor (lender).

The reader should note that the graph for the value of a financial transaction with a unique yield rate from the borrower's side of the transaction will be an increasing function rather than decreasing. In order to illustrate this, assume that there is only one party on the borrower's side of the transaction for the project summarized in Table 5.3. The corresponding graph for the borrower is given by the dashed line in Figure 5.6. Note that positive values of $P(i)$ lie to the right of the yield rate rather than to the left.

Example 5.12 Analyze the illustration given at the beginning of Section 5.3 as a capital budgeting exercise.

Figure 5.7 is a time diagram for this transaction. The two cash flows into the investment are shown at the top of the diagram and the one cash flow out of the investment is shown at the bottom.

Figure 5.7 Time diagram for Example 5.12

From formula (5.2)

$$P(i) = -100 + 230v - 132v^2$$

which is shown in Section 5.3 to have two yield rates, 10% and 20%. Table 5.4 tabulates $P(i)$ at a wide range of illustrative interest rates.

Table 5.4 Net Present Values for Example 5.12

Rate of interest i	*Net present value P(i)*
0%	- 2.00
5	- .68
10	0
15	+ .19
20	0
25	- .48

These results are displayed graphically in Figure 5.8. The maximum value of $P(i)$ occurs at 14.78%, but is equal to $P(i)$ at 15% to the two decimal places used in Table 5.4. The verification of this result is left as an exercise.

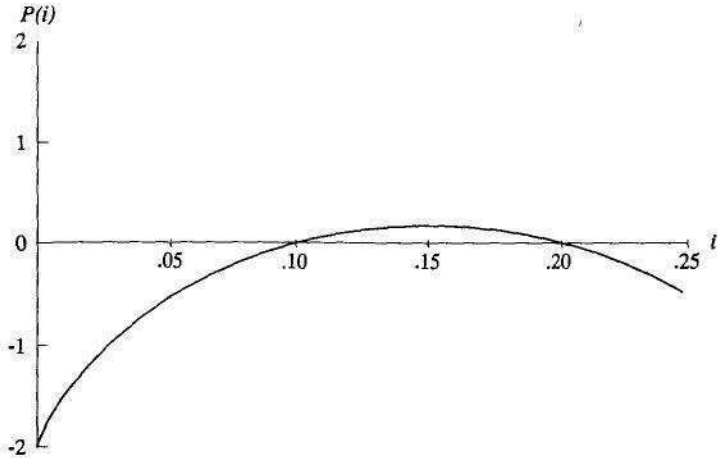

Figure 5.8 Net present values for Example 5.12

The possible non-uniqueness of the yield rate is often cited as a reason to favor the net present value method of capital budgeting over the yield rate method. However, we will now show that problems also exist in using the net present value method.

In Figure 5.8 $P(i)$ is a decreasing function in the vicinity of the 20% yield rate, which is the normal pattern from the lender's vantage point. However, $P(i)$ is an increasing function in the vicinity of the 10% yield rate. Thus, if the lender in this example requires only a 5% rate of return, the investment is a poor one since $P(i)$ is negative. However, if the lender triples the required rate of return to 15%, the investment somehow becomes a good one since $P(i)$ is now positve! This result is illogical, showing that using the net present value method of capital budgeting does not really solve the problems inherent when multiple yield rates exist.

5.9 MORE GENERAL BORROWING/LENDING MODELS

Sections 5.3 and 5.8 have illustrated some of the difficulties in finding rational interpretations of certain financial calculations and in making valid comparisons among different financial transactions when multiple yield rates exist. Various approaches have been suggested in practice to circumvent the problems inherent with multiple yield rates.

One such approach is to discount the future cash outflows at a prescribed rate of interest and then perform the rest of the calculations based only on the future cash inflows. The prescribed rate of interest is generally a conservative rate which the investor can earn with safety. In effect, the investor is "pre-funding" the future cash outflows, i.e. a fund equal to the present value of the future cash outflows computed at the prescribed rate could be established out of which such cash outflows could be paid. The yield rate now computed using

the future cash inflows will be unique. Whether or not such a fund is actually established by the investor is irrelevant to the validity of the calculation.

The remainder of this section discusses another approach which has been developed. In Section 5.3 we showed that if the outstanding investment balance is positive throughout the period of investment, then the yield rate will be unique. We can generalize this result and define a *pure investment project* as one in which $B_t \geq 0$ for $t = 0, 1, 2, \ldots, n$. A pure investment project is one in which the investor has money invested in the project throughout the period of investment.

We now switch to the perspective of a borrower and define a *pure financing project* as one in which $B_t \leq 0$ for $t = 0, 1, 2, \ldots, n$. A pure financing project is one in which the investor owes money to the project throughout the period of investment. Thus, the investor has actually become a borrower in this case.

Multiple yield rates can arise when some outstanding balances are positive and others are negative during the period of investment. We will call such a project a *mixed project*, since the investor is a lender during some portions of the period of investment and a borrower during other portions.

This more general model is based on the premise that a different rate of interest should be used during those portions of the period of investment during which the investor is in lender status than the rate used during those portions in borrower status. The *project return rate* is denoted by r and is the interest preference rate during those portions of the period of investment during which the investor is in lender status, i.e. the investment balance $B_t \geq 0$. The *project financing rate* is denoted by f and is the interest preference rate during those portions of the period of investment during which the investor is in borrower status, i.e. the investment balance $B_t < 0$.

Typically, r will be greater than f, since an astute investor will have a higher interest preference rate as a lender than as a borrower. However, the mathematical development does not require that $r > f$.

We generalize the approach used in developing formulas (5.4) through (5.6) to fit this situation. The initial fund balance is

$$B_0 = C_0. \tag{5.23}$$

Successive fund balances are developed as a recursion formula

$$B_t = B_{t-1}(1 + r) + C_t, \text{ if } B_{t-1} \geq 0 \tag{5.24a}$$

or

$$B_t = B_{t-1}(1 + f) + C_t, \text{ if } B_{t-1} < 0 \tag{5.24b}$$

for $t = 1, 2, \ldots, n$. The ending fund balance is a polynomial in r and f of the form

$$B_n = C_0(1+r)^{m_0}(1+f)^{n-m_0} + C_1(1+r)^{m_1}(1+f)^{n-m_1-1} + \cdots + C_n \qquad (5.25)$$

where the m_j's are integers such that $n \geq m_0 \geq m_1 \geq \ldots \geq m_n \geq 0$. In formula (5.25) m_j is the total number of periods from time j to time n for which interest rate r is used, with rate f being used for the remainder of the periods.

If $r = f$, formulas (5.23) through (5.25) simplify to formulas (5.4) through (5.6). Recall that $B_n = 0$ at the yield rate. If $r \neq f$, the concept of yield rate can still be utilized. However, in this more general case the yield rate is not a single number, but rather a functional relationship between r and f. In other words, for a given value of f, if a value of r can be found such that $B_n = 0$, then r and f are a yield rate pair for the transaction. A transaction for which yield rates exist will typically have an infinite number of r, f pairs and a functional relationship between r and f can be found.

The above constitutes a generalization of the yield rate method of capital budgeting. It is also possible to generalize the net present value method. Recall that formula (5.2) defining net present value is based on R_t's which are the negatives of the C_t's used above. Thus, positive values of net present value correspond to negative values of B_n, and conversely. The fact that negative values of B_n are favorable to the investor can be interpreted as reflecting the fact that a negative balance in the investment balance at the end of the investment period is, in effect, a positive balance to the investor.

Readers who are interested in a more extensive analysis of this borrowing/lending model are referred to two papers both by D. Teichroew, A. A. Robichek, and M. Montalbano (both 1965) listed in the bibliography.

***Example 5.13** An investor is required to make a contribution of $1600 immediately and $10,000 at the end of two years in exchange for receiving $10,000 at the end of one year.*

***(1)** Find the yield rates, if $r = f$.*

***(2)** Express r as a function of f, if r and f are a yield rate pair.*

***(3)** Would the investor accept or reject the transaction if $r = 70\%$ and $f = 30\%$?*

***(4)** Rework (3) if $f = 50\%$.*

1. Let $i = r = f$. The equation of value is

$$1600(1+i)^2 + 10{,}000 = 10{,}000(1+i).$$

This is a quadratic whose two roots are

$$i = .25, \text{ or } 25\%$$

and

$$i = 4, \text{ or } 400\%.$$

Thus, we have a transaction with multiple yield rates.

2. The investment balance is positive for the first year and negative for the second year. Thus, interest rate r is applicable for the first year and f is applicable for the second year. We have:

$$B_0 = 1600$$
$$B_1 = 1600(1 + r) - 10{,}000$$
$$B_2 = [1600(1 + r) - 10{,}000](1 + f) + 10{,}000 = 0.$$

This defines a functional relationship between r and f. Solving for r as a function of f

$$1 + r = \frac{10{,}000}{1600}\left[1 - \frac{1}{1+f}\right]$$
$$r = 6.25\left[1 - \frac{1}{1+f}\right] - 1$$
$$= 5.25 - \frac{6.25}{1+f}.$$

Some sample values of r and f are given in Table 5.5.

Table 5.5 Project Rates for Example 5.13

Financing rate f	*Return rate r*
25%	25%
100	212.5
150	275
200	317
300	369
400	400

Note that $r = f$ at the two yield rates which would be expected. Also, note that $r > f$ between the two yield rates, which is the normal relationship. However, outside this range $r < f$. Finally, note that r increases as f increases.

3. Using formulas (5.23) through (5.25) we have:

$$B_0 = C_o = 1600$$
$$B_1 = B_0(1 + r) + C_1$$
$$= 1600(1.7) - 10{,}000 = -7280$$
$$B_2 = B_1(1 + f) + C_2$$
$$= (-7280)(1.3) + 10{,}000 = 536.$$

Since $B_2 > 0$, the investor rejects this transaction.

4. Following the same approach used immediately above, B_0 and B_1 are unchanged. The new ending balance is

$$B_2 = (-7280)(1.5) + 10{,}000 = -920.$$

Since $B_2 < 0$, the investor accepts this transaction.

It is instructive to consider why (4) is accepted and (3) is rejected. The only difference between the two examples is in the borrowing interest preference rate f. If the maximum rate at which the investor is willing to borrow is 30%, the transaction should be rejected. However, if the investor is willing to borrow at rates up to 50%, then the transaction should be accepted.

EXERCISES

5.2 Discounted cash flow analysis

1. *a*) Express the present value of the contributions in Table 5.1 in terms of annuity-immediate symbols.
   b) Express the present value of the returns in Table 5.1 in terms of annuity-immediate symbols.

2. The internal rate of return for an investment in which $C_0 = \$3000$, $C_1 = \$1000$, $R_1 = \$2000$, and $R_2 = \$4000$ can be expressed as $1/n$. Find n.

3. A ten-year investment project requires an initial investment of \$100,000 at inception and maintenance expenses at the beginning of each year. The maintenance expense for the first year is \$3000, and is anticipated to increase 6% each year thereafter. Projected annual returns from the project are \$30,000 at the end of the first year decreasing 4% per year thereafter. Find R_6.

4. An investor enters into an agreement to contribute \$7000 immediately and \$1000 at the end of two years in exchange for the receipt of \$4000 at the end of one year and \$5500 at the end of three years. Find:
   a) $P(.09)$.
   b) $P(.10)$.

5. Find the Newton-Raphson iteration formula based on formula (5.2) which can be used in general to calculate an unknown yield rate.

5.3 Uniqueness of the yield rate

6. *a*) In Exercise 4 what is the maximum number of possible yield rates using Descartes' rule of signs?
   b) From the two answers it is apparent that a yield rate exists between 9% and 10%. Is this rate unique?
   c) Explain your answer to (*b*).

7. Payments of \$100 now and \$108.15 two years from now are equivalent to a payment of \$208 one year from now at either rate i or j. Find the absolute difference of the two rates.

8. Show that we cannot guarantee uniqueness of the yield rate if C_0 and C_n have the same sign.

9. An investor pays \$100 immediately and \$$X$ at the end of two years in exchange for \$200 at the end of one year. Find X such that two yield rates exist which are equal in absolute value but opposite in sign.

5.4 Reinvestment rates

10. It is desired to accumulate a fund of \$1000 at the end of 10 years by equal deposits at the beginning of each year. If the deposits earn interest at 8% effective but the interest can be reinvested at only 4% effective, show that the deposit necessary is

$$\frac{1000}{2s_{\overline{11}|.04} - 12}.$$

11. A loan of \$10,000 is being repaid with payments of \$1000 at the end of each year for 20 years. If each payment is immediately reinvested at 5% effective, find the effective annual rate of interest earned over the 20-year period.

12. An investor purchases a five-year financial instrument having the following features:
    (*i*) The investor receives payments of \$1000 at the end of each year for five years.
    (*ii*) These payments earn interest at an effective rate of 4% per annum. At the end of the year, this interest is reinvested at the effective rate of 3% per annum.

    Find the purchase price to the investor to produce a yield rate of 4%.

13. An investor deposits \$1000 at the beginning of each year for five years in a fund earning 5% effective. The interest from this fund can be reinvested at only 4% effective. Show that the total accumulated value at the end of ten years is

$$1250(s_{\overline{11}|.04} - s_{\overline{6}|.04} - 1).$$

14. A invests \$2000 at an effective interest rate of 17% for 10 years. Interest is payable annually and is reinvested at an effective rate of 11%. At the end of 10 years, the accumulated interest is \$5685.48. B invests \$150 at the end of each year for 20 years at an effective interest rate of 14%. Interest is payable annually and is reinvested at an effective rate of 11%. Find B's accumulated interest at the end of 20 years.

5.5 Interest measurement of a fund

15. The funds of a firm at the beginning of the year were \$500,000 and at the end of the year \$680,000. Gross interest earned was \$60,000, against which there were

investment expenses of \$5000. Find the net effective rate of interest yielded by the funds.

16. A fund earning 4% effective has a balance of \$1000 at the beginning of the year. If \$200 is added to the fund at the end of three months and if \$300 is withdrawn from the fund at the end of nine months, find the ending balance under the assumption
$$_{1-t}i_t = (1-t)i.$$

17. *a*) Under the assumption that $_{1-t}i_t = (1-t)i$, find an expression for $_ti_0$.
b) Under the assumption that $_ti_0 = ti$, find an expression for $_{1-t}i_t$.

18. Let the exposure associated with i (i.e. the denominator of formula (5.13)) be denoted by E. By using formula (5.17) show that the force of interest at any point is given by
$$\delta_t = \frac{I}{E + (1-t)I}.$$
It should be noted that in terms of the time diagram, this is equivalent to bringing the entire amount of interest I onto the time diagram at time t instead of time 1. The expression $E + (1-t)I$ is often called the "exposure associated with δ_t."

19. Derive formula (5.20).

20. Find an expression for the total net contributions during the n periods in formula (5.18).

5.6 Time-weighted rates of interest

21. In Example 5.9 assume that May 1 is changed to June 1 and November 1 is changed to October 1.
a) Would the yield rate change when computed by the dollar-weighted method?
b) Would the yield rate change when computed by the time-weighted method?

22. In Example 5.9 assume that everything is unchanged except that an additional \$5000 is withdrawn on July 1.
a) Recompute the yield rate by the dollar-weighted method.
b) The yield rate by the time-weighted method cannot be calculated. Explain why not.

23. Deposits of \$1000 are made into an investment fund at time 0 and time 1. The fund balance is \$1200 at time 1 and \$2200 at time 2.
a) Compute the annual effective yield rate computed by a dollar-weighted calculation.
b) Compute the annual effective yield rate which is equivalent to that produced by a time-weighted calculation.

24. Let A be the fund balance on January 1, B the balance on June 30, and C the balance on December 31.

a) If there are no deposits or withdrawals, show that yield rates computed by the dollar-weighted method and the time-weighted method are both equal to $(C - A)/A$.

b) If there was a single deposit of D immediately after the June 30 balance was calculated, find expressions for the dollar-weighted and time-weighted yield rates.

c) Rework (*b*) if the deposit occurred immediately before the June 30 balance was calculated.

d) Verbally interpret the fact that the dollar-weighted yield rates in (*b*) and (*c*) are equal.

e) Show that the time-weighted yield rate in (*b*) is greater than in (*c*).

5.7 Portfolio methods and investment year methods

25. Find $\ddot{s}_{\overline{5}|}$ from the data in Table 5.2 assuming the first payment is made in calendar year $z + 3$.

26. It is known that $1 + i_t^y = (1.08 + .005t)^{1+.01y}$ for integral $t, 1 \le t \le 5$, and integral $y, 0 \le y \le 10$. If \$1000 is invested for three years beginning in year $y = 5$, find the equivalent level effective rate of interest.

27. An investment year method is defined by creating an accumulation function which is a function of two variables. Let $a(s, t)$ be the accumulated value at time t of an original investment of one unit at time s, where $0 \le s \le t$.

a) Express $\delta_{s,t}$ in terms of $a(s, t)$.

b) Express $a(s, t)$ in terms of $\delta_{s,t}$.

c) Express $a(s, t)$ in terms of $a(s)$ and $a(t)$ for the portfolio method.

d) Find $a(0, t)$ assuming a level effective rate of interest i.

e) Find $a(t, t)$.

5.8 Capital budgeting

28. If an investor's interest preference rate is 12%, should the investment in Exercise 4 be accepted or rejected?

29. A used car can be purchased for \$5000 cash or for \$2400 down and \$1500 at the end of each of the next two years. Should a purchaser with an interest preference rate of 10% pay cash or finance the car?

30. *a*) Draw a graph of $P(i)$ for Example 5.4.

b) What can you conclude about the graph of $P(i)$ when all yield rates are imaginary?

5.9 More general borrowing/lending models

31. An investor considering whether or not to make the investment given in the illustration at the beginning of Section 5.3 is confused that the yield rate is equal to

either 10% or 20%. In order to resolve the confusion, the investor decides to "pre-fund" the contribution at the end of two years in a fund earning 12%.

a) Compute the yield rate on the transaction under these conditions.

b) Is the yield rate in (*a*) unique?

32. Again considering the same transaction as in Exercise 31, find the project financing rate f corresponding to a project return rate $r = 15\%$.

33. The cash flows for the first five years of a ten-year investment period are as follows:

t	C_t
0	1000
1	2000
2	-4000
3	3000
4	-4000
5	5000

If $r = 15\%$ and $f = 10\%$, find B_5.

Miscellaneous problems

34. An investment account is established on which it is estimated that 8% can be earned over the next 20 years. If the interest each year is subject to income tax at a 25% tax rate, find the percentage reduction in the accumulated interest at the end of 20 years.

35. A borrower needs \$800. The funds can be obtained in two ways:

(*i*) By promising to pay \$900 at the end of the period.

(*ii*) By borrowing \$1000 and repaying \$1120 at the end of the period.

If the interest preference rate for the period is 10%, which option should be chosen?

36. The proceeds from a life insurance policy are left on deposit, with interest credited at the end of each year. The beneficiary makes withdrawals from the fund at the end of each year$t$, for $t = 1, 2, \ldots, 10$. At the minimum interest rate of 3% guaranteed in the policy, the equal annual withdrawal would be \$1000. However, the insurer credits interest at the rate of 4% for the first four years and 5% for the next six years. The actual amount withdrawn at the end of year t is

$$W_t = \frac{F_t}{\ddot{a}_{\overline{11-t}|.03}}$$

where F_t is the amount of the fund, including interest, prior to the withdrawal. Calculate W_{10}.

37. An investment fund had a balance on January 1 of \$273,000 and a balance on December 31 of \$372,000. The amount of interest earned during the year was

$18,000 and the computed yield rate on the fund was 6%. What was the average date for contributions to and withdrawals from the fund?

38. An investment fund is started with an initial deposit of 1 at time 0. New deposits are made continuously at the annual rate $1 + t$ at time t over the next n years. The force of interest at time t is given by $\delta_t = (1 + t)^{-1}$. Find the accumulated value in the fund at the end of n years.

6

Amortization schedules and sinking funds

6.1 INTRODUCTION

In Chapter 6 various methods of repaying a loan are analyzed in more depth than in previous chapters. In particular, two methods of repaying a loan are discussed:

1. *The amortization method.* In this method the borrower repays the lender by means of installment payments at periodic intervals. This process is called "amortization" of the loan.
2. *The sinking fund method.* In this method the borrower repays the lender by means of one lump-sum payment at the end of the term of the loan. The borrower pays interest on the loan in installments over this period. It is also assumed that the borrower makes periodic payments into a fund, called a "sinking fund," which will accumulate to the amount of the loan to be repaid at the end of the term of the loan.

Chapter 6 also considers the following questions which are related to the repayment methods mentioned above:

1. How can the outstanding loan balance at any given point in time be determined?
2. How can any payments made by the borrower be divided into repayment of principal and payment of interest?

Sections 6.2 and 6.3 consider the amortization method, while Section 6.4 considers the sinking fund method. Succeeding sections develop extensions and generalization of both methods.

6.2 FINDING THE OUTSTANDING LOAN BALANCE

If a loan is being repaid by the amortization method, the installment payments form an annuity whose present value is equal to the original amount of the loan. Section 6.2 is concerned with determining the amount of the outstanding loan balance any time after the inception date of the loan. The reader may encounter a variety of terms synonymous with "outstanding loan balance" in common usage. Among these are "outstanding principal," "unpaid balance," and "remaining loan indebtedness."

Determining the amount of the outstanding loan balance is of great significance in practice. For example, if a family is buying a home with a 30-year mortgage, after making mortgage payments for 12 years, how much would they have to pay in one lump sum in order to completely pay off the mortgage?

These are two approaches used in finding the amount of the outstanding loan balance; namely, the prospective method and the retrospective method. The names chosen are appropriate, since the prospective method calculates the outstanding loan balance looking into the future, while the retrospective method calculates the outstanding loan balance looking into the past.

According to the *prospective method*, the outstanding loan balance at any point in time is equal to the present value at that date of the remaining payments. According to the *retrospective method*, the outstanding loan balance at any point in time is equal to the original amount of the loan accumulated to that date less the accumulated value at that date of all payments previously made.

It is possible to show that, in general, the prospective and retrospective methods are equivalent. At the inception date of the loan we have the following equality:

$$\text{Present Value of Payments} = \text{Amount of Loan.}$$

We now accumulate each side of this equation to the date at which the outstanding loan balance is desired, obtaining:

$$\text{Current Value of Payments} = \text{Accumulated Value of Loan.}$$

However, the payments can be divided into past and future payments giving:

$$\text{Accumulated Value of Past Payments} + \text{Present Value of Future Payments} = \text{Accumulated Value of Loan.}$$

Now, rearranging, we obtain:

Present Value of Future Payments = Accumulated Value of Loan − Accumulated Value of Past Payments,

or,

Prospective Method = Retrospective Method.

We will use the notation developed in Chapter 5 and denote the outstanding loan balance at time t by the symbol B_t. As an aid in using this symbol, we can denote the prospective form by B_t^p and the retrospective form by B_t^r. However, the usage of the superscripts p and r should be considered optional. The original loan balance B_0 is often denoted by L.

In specific cases it is possible to prove that the prospective method is equal to the retrospective method algebraically. For example, consider a loan of $a_{\overline{n}|}$ at interest rate i per period being repaid with payments of 1 at the end of each period for n periods, for which the outstanding loan balance t periods after the inception date of the loan is desired where $0 < t < n$. The outstanding loan balance at time t is computed after making the tth payment.

The prospective method gives

$$B_t^p = a_{\overline{n-t}|}. \tag{6.1}$$

The retrospective method gives

$$B_t^r = a_{\overline{n}|}(1+i)^t - s_{\overline{t}|}. \tag{6.2}$$

We can show that the retrospective form is equal to the prospective form:

$$\begin{aligned} a_{\overline{n}|}(1+i)^t - s_{\overline{t}|} &= \frac{1-v^n}{i}(1+i)^t - \frac{(1+i)^t - 1}{i} \\ &= \frac{(1+i)^t - v^{n-t} - (1+i)^t + 1}{i} \\ &= \frac{1-v^{n-t}}{i} \\ &= a_{\overline{n-t}|}. \end{aligned}$$

In any given problem the prospective method or the retrospective method may be more efficient depending upon the nature of the problem. If the size and number of payments are known, then the prospective method is usually more efficient. If the number of payments or the size of a final irregular payment is not known, then the retrospective method is usually more efficient.

***Example 6.1** A loan is being repaid with 10 payments of $2000 followed by 10 payments of $1000 at the end of each half-year. If the nominal rate of interest convertible semiannually is 10%, find the outstanding loan balance*

immediately after five payments have been made by both the prospective method and the retrospective method.

1. The rate of interest is 5% per half-year. Prospectively, the outstanding loan balance is

$$B_5^p = 1000(a_{\overline{15}|} + a_{\overline{5}|}) = 1000(10.3797 + 4.3295) = \$14{,}709$$

to the nearest dollar.

2. The original loan was

$$L = 1000(a_{\overline{20}|} + a_{\overline{10}|}) = 1000(12.4622 + 7.7217) = \$20{,}184$$

to the nearest dollar. Retrospectively, the outstanding loan balance is

$$B_5^r = 20{,}184(1.05)^5 - 2000s_{\overline{5}|} = 20{,}184(1.27628) - 2000(5.5256) = \$14{,}709$$

to the nearest dollar. Thus, the prospective and retrospective methods produce the same answer.

Example 6.2 A loan is being repaid with 20 annual payments of \$1000 each. At the time of the fifth payment, the borrower wishes to pay an extra \$2000 and then repay the balance over 12 years with a revised annual payment. If the effective rate of interest is 9%, find the amount of the revised annual payment.

The balance after five years, prospectively, is

$$B_5^p = 1000a_{\overline{15}|} = 1000(8.0607) = \$8060.70.$$

If the borrower pays an additional \$2000, the balance becomes \$6060.70. An equation of value for the revised payment, denoted by X, is

$$Xa_{\overline{12}|} = 6060.70$$

or

$$X = \frac{6060.70}{7.1607} = \$846.38.$$

6.3 AMORTIZATION SCHEDULES

If a loan is being repaid by the amortization method, each payment is partially repayment of principal and partially payment of interest. Section 6.3 is concerned with determining how each payment can be divided between principal and interest.

Determining the amount of principal and interest contained in each payment is important for both the borrower and lender. For example, principal and interest are generally treated quite differently for income tax purposes.

An *amortization schedule* is a table which shows the division of each payment into principal and interest, together with the outstanding loan balance

after each payment is made. Consider a loan of $a_{\overline{n}|}$ at interest rate i per period being repaid with payments of 1 at the end of each period for n periods. Table 6.1 is an amortization schedule for this case.

Table 6.1 Amortization Schedule for a Loan of $a_{\overline{n}|}$ Repaid Over n Periods at Rate i.

Period	*Payment amount*	*Interest paid*	*Principal repaid*	*Outstanding loan balance*
0				$a_{\overline{n}|}$
1	1	$ia_{\overline{n}|} = 1 - v^n$	v^n	$a_{\overline{n}|} - v^n = a_{\overline{n-1}|}$
2	1	$ia_{\overline{n-1}|} = 1 - v^{n-1}$	v^{n-1}	$a_{\overline{n-1}|} - v^{n-1} = a_{\overline{n-2}|}$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
t	1	$ia_{\overline{n-t+1}|} = 1 - v^{n-t+1}$	v^{n-t+1}	$a_{\overline{n-t+1}|} - v^{n-t+1} = a_{\overline{n-t}|}$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$n-1$	1	$ia_{\overline{2}|} = 1 - v^2$	v^2	$a_{\overline{2}|} - v^2 = a_{\overline{1}|}$
n	1	$ia_{\overline{1}|} = 1 - v$	v	$a_{\overline{1}|} - v = 0$
Total	n	$n - a_{\overline{n}|}$	$a_{\overline{n}|}$	

Consider the first period of the loan. At the end of the first period the interest due on the balance at the beginning of the period is $ia_{\overline{n}|} = 1 - v^n$. The rest of the total payment of 1, i.e. v^n, must be principal repaid. The outstanding loan balance at the end of the period equals the outstanding loan balance at the beginning of the period less the principal repaid, i.e. $a_{\overline{n}|} - v^n = a_{\overline{n-1}|}$. The same reasoning applies for each successive line of the schedule.

Several additional observations are possible. First, it should be noted that the outstanding loan balance agrees with that obtained by the prospective method in formula (6.1.). Second, the sum of the principal repayments equals the original amount of the loan. Third, the sum of the interest payments is equal to the difference between the sum of the total payments and the principal repayments. Fourth, the principal repayments form a geometric progression with common ratio $1 + i$. Thus, it is a simple matter to find any one principal repayment knowing any other principal repayment and the rate of interest.

Further insight into the nature of the amortization schedule can be gained by the following argument. The original loan balance $a_{\overline{n}|}$ will accumulate to $a_{\overline{n}|}(1 + i) = \ddot{a}_{\overline{n}|}$ at the end of the first period. However, $\ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$,

i.e. $\ddot{a}_{\overline{n}|}$ is sufficient to make the annuity payment of 1 and leave an outstanding balance of $a_{\overline{n-1}|}$ at the end of the first period. The same reasoning applies for each successive line of the schedule.

For convenience, we will denote the amount of interest paid in the tth installment by I_t and the amount of principal repaid in the same installment by P_t. Thus, for the amortization schedule given in Table 6.1 we have

$$I_t = 1 - v^{n-t+1} \tag{6.3}$$

and

$$P_t = v^{n-t+1}. \tag{6.4}$$

The reader should note that the notation I_t was also used in Chapter 1 in a different context. However, in both cases the symbol refers to an amount of interest earned between times $t - 1$ and t, so there should be little risk of ambiguity. Finally, we will let the installment payment be denoted by R.

It should be noted that Table 6.1 is based on an original loan of $a_{\overline{n}|}$. If the original loan were some other amount, then all the values in the schedule would be proportional. For example, if the original loan were \$1000, then each number in the last four columns of the schedule would be multiplied by $1000/a_{\overline{n}|}$.

For a specific problem, the amortization schedule can be constructed from basic principles. For example, consider the construction of an amortization schedule for a \$1000 loan repaid in four annual payments if the annual effective rate of interest is 8%. Then we have

$$R = \frac{1000}{a_{\overline{4}|}} = \frac{1000}{3.3121} = \$301.92.$$

Table 6.2 is the amortization schedule for this example.

Table 6.2 Amortization Schedule for a Loan of \$1000 Repaid Over Four Years at 8%

Year	*Payment amount*	*Interest paid*	*Principal repaid*	*Outstanding loan balance*
0				1000.00
1	301.92	80.00	221.92	778.08
2	301.92	62.25	239.67	538.41
3	301.92	43.07	258.85	279.56
4	301.92	22.36	279.56	0

In the first line of Table 6.2 we have the following calculations. The interest contained in the first payment is

$$I_1 = i \cdot B_0 = .08(1000) = \$80.00.$$

The principal contained in the first payment is

$$P_1 = R - I_1 = 301.92 - 80.00 = \$221.92.$$

The outstanding loan balance at the end of the first year is

$$B_1 = B_0 - P_1 = 1000.00 - 221.92 = \$778.08.$$

Each successive line of the schedule is calculated in similar fashion.

In this particular example the last line exactly balances. However, more typically some roundoff error is likely to accumulate. Standard practice is to adjust the last payment so that it is exactly equal to the amount of interest for the final period plus the outstanding loan balance at the beginning of the final period (i.e. at the end of the next-to-final period). This adjustment will bring the outstanding loan balance exactly to zero at the end of the term of the loan.

It is also possible to construct the amortization schedule by alternate methods utilizing the various relationships in the table. As one example, the values of the outstanding loan balance can be calculated as in Section 6.2, and then the rest of the schedule can be deduced from these values. As a second example, the principal repaid column can be calculated using the fact that the successive values are in geometric progression, and then the rest of the schedule can be deduced from these values.

It should be noted that if it is desired to find the amount of principal and interest in one particular payment, it is not necessary to construct the entire amortization schedule. The outstanding loan balance at the beginning of the period in question can be determined by the methods of Section 6.2, and then that one line of the amortization schedule can be calculated.

The reader might wonder about the amortization schedule for a perpetuity. It is clear that for a perpetuity the entire payment represents interest and the outstanding loan balance remains unchanged. Thus, amortization of a perpetuity is really a contradiction in terms.

Several assumptions have been implicit in the preceding discussion of amortization schedules. First, we have assumed a constant rate of interest. An amortization schedule at a fixed, but varying rate of interest will be considered in the exercises. Second, we have assumed that the annuity payment period and the interest conversion period are equal. The cases in which they are not equal will be considered in Section 6.5. Third, we have assumed that annuity payments are level. The situation in which they are level except for a final

irregular payment will be considered in Example 6.3 and in the exercises. The general case of a varying series of payments will be considered in Section 6.6.

Certain special types of loan transactions are discussed in more detail in Chapter 8. In particular, real estate mortgages are considered in Section 8.3. One important new type of mortgage considered in Section 8.3 is the adjustable rate mortgage. This type of mortgage involves features not considered in Chapter 6.

Example 6.3 A $1000 loan is being repaid by payments of $100 at the end of each quarter for as long as necessary, plus a smaller final payment. If the nominal rate of interest convertible quarterly is 16%, find the amount of principal and interest in the fourth payment.

The outstanding loan balance at the beginning of the fourth quarter, i.e. the end of the third quarter, is

$$B_4^r = 1000(1.04)^3 - 100s_{\overline{3}|} = 1124.86 - 312.16 = \$812.70.$$

The interest contained in the fourth payment is

$$I_4 = .04(812.70) = \$32.51.$$

The principal contained in the fourth payment is

$$P_4 = 100.00 - 32.51 = \$67.49.$$

Note that it was not necessary to find the duration and amount of the smaller final payment in order to solve this example.

Example 6.4 A borrows $10,000 from B and agrees to repay it with equal quarterly installments of principal and interest at 8% convertible quarterly over six years. At the end of two years B sells the right to receive future payments to C at a price which produces a yield rate of 10% convertible quarterly. Find the total amount of interest received: (1) by C, and (2) by B.

The quarterly installment paid by A is

$$\frac{10{,}000}{a_{\overline{24}|.02}} = \frac{10{,}000}{18.9139} = \$528.71.$$

The price C pays is then

$$528.71a_{\overline{16}|.025} = (528.71)(13.0550) = \$6902.31.$$

The total payments made by A over the last four years are

$$(16)(528.71) = \$8459.36.$$

Thus, the total interest received by C is

$$8459.36 - 6902.31 = \$1557.05.$$

2. The outstanding loan balance on B's original amortization schedule at the end of two years is

$$528.71a_{\overline{16}|.02} = (528.71)(13.5777) = \$7178.67.$$

The total payments made by A over the first two years are

$$(8)(528.71) = \$4229.68.$$

The total principal repaid by A over this period is

$$10{,}000 - 7178.67 = \$2821.33.$$

Thus, the total interest received by B apparently is

$$4229.68 - 2821.33 = \$1408.35.$$

However, the total interest paid by A over the entire loan is equal to

$$(24)(528.71) - 10{,}000 = \$2689.04$$

which does not equal the sum of the interest received by C and B, i.e.

$$1557.05 + 1408.35 = \$2965.40.$$

What has happened is that B incurred a loss at the end of two years equal to the difference between the outstanding loan balance and the price to C, i.e.

$$7178.67 - 6902.31 = \$276.36.$$

If this loss is offset against the interest received by B, then B's net investment income from this transaction is

$$1408.35 - 276.36 = \$1131.99.$$

The system now balances, since adding this amount to the amount of interest received by C is equal to the total interest paid by A, i.e.

$$1557.05 + 1131.99 = \$2689.04.$$

***Example 6.5** An amount is invested at an annual effective rate of interest i which is just sufficient to pay 1 at the end of each year for n years. In the first year the fund actually earns rate i and 1 is paid at the end of the year. However, in the second year the fund earns rate j where $j > i$. Find the revised payment which could be made at the ends of years 2 through n: (1) assuming the rate earned reverts back to i again after this one year, and (2) assuming the rate earned remains at j for the rest of the n-year period.*

1. The initial investment is $B_0 = a_{\overline{n}|i}$ and the account balance at the end of the first year is $B_1 = a_{\overline{n-1}|i}$. Let X be the revised payment. We now construct the next line of the amortization schedule on the revised basis, obtaining

$$I_2 = ja_{\overline{n-1}|i}$$
$$P_2 = X - ja_{\overline{n-1}|i}$$

and

$$\begin{aligned} B_2 &= a_{\overline{n-1}|i} - (X - ja_{\overline{n-1}|i}) \\ &= (1+j)a_{\overline{n-1}|i} - X. \end{aligned}$$

However, prospectively B_2 must equal the present value of the future payments. Thus, we have

$$\begin{aligned} (1+j)a_{\overline{n-1}|i} - X &= Xa_{\overline{n-2}|i} \\ X(1 + a_{\overline{n-2}|i}) &= (1+j)a_{\overline{n-1}|i} \\ X\ddot{a}_{\overline{n-1}|i} &= (1+j)a_{\overline{n-1}|i} \\ X(1+i)a_{\overline{n-1}|i} &= (1+j)a_{\overline{n-1}|i} \end{aligned}$$

which gives

$$X = \frac{1+j}{1+i}.$$

2. The development is identical to case 1 above, except that the prospective value for the future payments, which equals B_2, is computed at rate j instead of i. Thus, we have

$$\begin{aligned} (1+j)a_{\overline{n-1}|i} - X &= Xa_{\overline{n-2}|j} \\ X(1 + a_{\overline{n-2}|j}) &= (1+j)a_{\overline{n-1}|i} \\ X\ddot{a}_{\overline{n-1}|j} &= (1+j)a_{\overline{n-1}|i} \\ X(1+j)a_{\overline{n-1}|j} &= (1+j)a_{\overline{n-1}|i} \end{aligned}$$

which gives

$$X = \frac{a_{\overline{n-1}|i}}{a_{\overline{n-1}|j}}.$$

6.4 SINKING FUNDS

Rather than repay a loan in installments by the amortization method, a borrower may choose to repay it by means of one lump-sum payment at the end of a specified period of time. In many such cases the borrower will accumulate a fund which will be sufficient to exactly repay the loan at the end of the specified period of time. In fact, in some cases the lender will insist that the borrower accumulate such a fund, which is often called a *sinking fund.*

In some cases payments into a sinking fund may vary irregularly at the discretion of the borrower. However, we shall be primarily interested in those cases in which the payments follow a regular pattern, i.e. where they are some form of an annuity.

It is usually required that the borrower pay interest on the loan periodically over the term of the loans. Such interest is sometimes called *service* on the loan. Thus, the amount of the loan remains constant.

Since the balance in the sinking fund at any point could presumably be applied against the loan, the net amount of the loan is equal to the original amount of the loan minus the accumulated value of the sinking fund. This

concept plays the same role for the sinking fund method that the outstanding loan balance, discussed in Section 6.2, does for the amortization method.

It is possible to show that if the rate of interest paid on the loan equals the rate of interest earned on the sinking fund, then the sinking fund method is equivalent to the amortization method.

Recall formula (3.6)

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{s_{\overline{n}|}} + i. \tag{3.6}$$

Consider a loan of amount 1 repaid over n periods. The expression $1/a_{\overline{n}|}$ is the amount of each payment necessary to repay the loan by the amortization method. However, the expression $1/s_{\overline{n}|}$ is the periodic sinking fund deposit necessary to accumulate the amount of the loan at the end of n periods, while i is the amount of interest paid on the loan each period. Thus, the two methods are equivalent.

It is instructive to consider this equivalence from an alternate viewpoint. Consider a loan of amount $a_{\overline{n}|}$ being repaid with installments of 1 at the end of each period for n periods. The amount of interest each period is $ia_{\overline{n}|}$. Thus, $1 - ia_{\overline{n}|}$ is left to go into the sinking fund each period. However, the sinking fund will accumulate to

$$(1 - ia_{\overline{n}|})s_{\overline{n}|} = v^n s_{\overline{n}|} = a_{\overline{n}|}$$

which is the original amount of the loan.

At first glance it might seem that the two methods cannot be equivalent, since from Table 6.1, the interest paid in the amortization method decreases $1 - v^n, 1 - v^{n-1}, \ldots, 1 - v$; while the interest paid in the sinking fund method is a constant $ia_{\overline{n}|} = 1 - v^n$ each period. However, each period the sinking fund earns interest which exactly offsets the seeming discrepancy, so that the net amount of interest is the same for the sinking fund method as for the amortization method.

For example, during the tth period, $1 \leq t \leq n$, the amount of interest in the amortization schedule is

$$ia_{\overline{n-t+1}|} = 1 - v^{n-t+1}.$$

The net amount of interest in the sinking fund method is the amount of interest paid $ia_{\overline{n}|}$ less the amount of interest earned on the sinking fund. The amount in the sinking fund is the accumulated value of the sinking fund deposits of $1 - ia_{\overline{n}|}$ at the end of $t - 1$ periods, i.e.

$$(1 - ia_{\overline{n}|})s_{\overline{t-1}|}.$$

Thus, the net amount of interest in the sinking fund method in the tth period is

$$\begin{aligned} ia_{\overline{n}|} - i(1 - ia_{\overline{n}|})\, s_{\overline{t-1}|} &= (1 - v^n) - v^n\left[(1+i)^{t-1} - 1\right] \\ &= 1 - v^n - v^{n-t+1} + v^n \\ &= 1 - v^{n-t+1}. \end{aligned}$$

Therefore, the net amount of interest in the sinking fund method is equal to the amount of interest in the amortization method if the rate of interest on the loan equals the rate of interest earned on the sinking fund.

The equivalence in the methods can be seen from a consideration of a *sinking fund schedule*. Table 6.3 is a sinking fund schedule for the same example considered in Table 6.2. Let the sinking fund deposit be denoted by D. Then we have

$$D = \frac{1000}{s_{\overline{4}|}} = \frac{1000}{4.5061} = \$221.92.$$

The reader should verify the entries in Table 6.3.

Table 6.3 Sinking Fund Schedule for a Loan of $1000 Repaid Over Four Years at 8%

Year	*Interest paid*	*Sinking fund deposit*	*Interest earned on sinking fund*	*Amount in sinking fund*	*Net amount of loan*
0					1000.00
1	80.00	221.92	0	221.92	778.08
2	80.00	221.92	17.75	461.59	538.41
3	80.00	221.92	36.93	720.44	279.56
4	80.00	221.92	57.64	1000.00	0

The following relationships between Table 6.2 and Table 6.3 should be noted:

1. The total payment in the sinking fund method, i.e. interest paid on the loan plus the sinking fund deposit, equals the payment amount in the amortization method.
2. The net interest paid in the sinking fund method, i.e. interest paid on the loan minus interest earned on the sinking fund, equals the interest paid in the amortization method.
3. The annual increment in the sinking fund, i.e. the sinking fund deposit plus the interest earned on the sinking fund, equals the principal repaid in the amortization method.

4. The net amount of the loan in the sinking fund method, i.e. the original amount of the loan minus the amount in the sinking fund, equals the outstanding loan balance in the amortization method.

It now remains to consider the operation of the sinking fund method when the rate of interest earned on the sinking fund differs from the rate of interest paid on the loan. We will denote the rate of interest paid on the loan by i and the rate of interest earned on the sinking fund by j.

In practice j is usually less than or equal to i. It would be unusual for a borrower to be able to accumulate money in a sinking fund at a higher rate of interest than is being paid on the loan. However, this is not necessarily the case mathematically, and the following analysis is valid even if j is greater than i.

The same basic approach will be used for the case in which $i \neq j$ as was previously used in the case in which the two rates were equal. The total payment will be split into two parts. First, interest at rate i will be paid on the amount of the loan. Second, the remainder of the total payment not needed for interest will be placed into a sinking fund accumulating at rate j.

Let $a_{\overline{n}|i\,\&\,j}$ represent the present value of an annuity of 1 at the end of each period for n periods under the conditions just described. Then if a loan of 1 is made, the periodic installment under the amortization method will be $1/a_{\overline{n}|i\,\&\,j}$. However, from the sinking fund method this payment must pay interest at rate i on the loan and provide for a sinking fund deposit which will accumulate at rate j to the amount of the loan at the end of n periods. Thus

$$\frac{1}{a_{\overline{n}|i\,\&\,j}} = \frac{1}{s_{\overline{n}|j}} + i\,. \tag{6.5}$$

We can now find an expression for $a_{\overline{n}|i\,\&\,j}$ as follows:

$$\begin{aligned}\frac{1}{a_{\overline{n}|i\,\&\,j}} &= \frac{1}{s_{\overline{n}|j}} + i \\ &= \frac{1}{a_{\overline{n}|j}} + (i - j)\,. \end{aligned} \tag{6.6}$$

Thus,

$$a_{\overline{n}|i\,\&\,j} = \frac{a_{\overline{n}|j}}{1 + (i - j)a_{\overline{n}|j}}\,. \tag{6.7}$$

It should be noted that if $i = j$, then $a_{\overline{n}|i\,\&\,j} = a_{\overline{n}|i}$ as would be expected.

The construction of a sinking fund schedule at two rates of interest is very similar to the construction of a sinking fund schedule at a single rate of interest. As an example, consider a \$1000 loan for four years on which an annual effective rate of interest of 10% is charged if the borrower accumulates the amount necessary to repay the loan by means of four annual sinking fund

deposits in a fund earning an annual effective rate of 8%. The total semiannual payment made by the borrower is

$$\frac{1000}{a_{\overline{4}|.10\,\&\,.08}} = \frac{1000}{s_{\overline{4}|.08}} + 1000(.10) = 221.92 + 100.00 = \$321.92.$$

This example is a generalization of the example considered in Table 6.3. Note that the sinking fund schedule is identical to Table 6.3, except that each entry in the interest paid column is \$100 instead of \$80.

In general, the sinking fund schedule at two rates of interest is identical to the sinking fund schedule at one rate of interest equal to the rate of interest earned on the sinking fund, except that a constant addition of $(i - j)$ times the amount of the loan is added to the interest paid column.

***Example 6.6** A wishes to borrow \$1000. Lender B offers a loan in which the principal is to be repaid at the end of four years. In the meantime 10% effective is to be paid on the loan and A is to accumulate the amount necessary to repay the loan by means of annual deposits in a sinking fund earning 8% effective. Lender C offers a loan for four years in which A repays the loan by the amortization method. What is the largest effective rate of interest that C can charge so that A is indifferent between the two offers?*

Under either method A will make four equal payments at the end of each year to repay the loan. Thus, A will be indifferent between the two offers if the annual payment is equal on both of them.

On the sinking fund offer from B the annual payment is

$$\frac{1000}{a_{\overline{4}|.10\,\&\,.08}} = \$321.92$$

as shown earlier in this section. Thus, on the amortization offer, C could charge i where

$$321.92\, a_{\overline{4}|\,i} = 1000$$

or

$$a_{\overline{4}|} = 3.1064.$$

We use the techniques of Section 3.8 to solve for the unknown rate of interest on the annuity by iteration. Formula (3.29) is used to obtain the starting value for the iteration

$$i_0 = \frac{2(n-k)}{k(n+1)} = \frac{2(4 - 3.1064)}{(3.1064)(5)} = .1151.$$

Successive applications of the Newton-Raphson iteration formula (3.28) produce

$$\begin{aligned} i_1 &= .1093 \\ i_2 &= .1094 \\ i_3 &= .1094. \end{aligned}$$

Thus, C could charge an effective rate of 10.94%.

In essence we have shown that

$$a_{\overline{4}|\ .10\,\&\,.08} = a_{\overline{4}|\ .1094}.$$

This example illustrates that to a borrower the total periodic payment is of primary concern and that the difference between the amortization method and the sinking fund method is somewhat artificial.

The reader may find the answer of .1094 surprising, since the answer might be expected to lie between .08 and .10. In general, if the equivalent rate of interest in the amortization method is denoted by i', we have the following approximate equality:

$$i' \approx i + \tfrac{1}{2}(i - j).$$

This formula would produce an answer for this example equal to $.10 + \frac{1}{2}(.10 - .08) = .11$, which is close to the true answer of .1094.

The equivalent rate i' is greater than i because the borrower not only is paying i per unit borrowed but also is investing in a sinking fund on which interest is being sacrificed at rate $i - j$. Since the average balance in the sinking fund per unit borrowed is 1/2, the extra interest cost is approximately $\frac{1}{2}(i - j)$. Thus, the total interest cost per unit borrowed is approximately

$$i + \tfrac{1}{2}(i - j).$$

It is also instructive to consider B's yield rate on this transaction. If A invests the sinking fund with B, then B's yield rate over the four-year term of the loan is the same as the cost to A, i.e. 10.94%. However, if A invests the sinking fund elsewhere, then B's yield rate is just 10% (ignoring reinvestment rates). Thus, B's overall yield rate is affected by whether or not B has the advantage of holding the sinking fund on which only 8% is credited.

***Example 6.7** An investor buys an n-year annuity with a present value of $1000 at 8% at a price which will permit the replacement of the original investment in a sinking fund earning 7% and will produce an overall yield rate of 9%. Find the purchase price of the annuity.*

The annual payment of the annuity is

$$\frac{1000}{a_{\overline{n}|.08}}.$$

Let P be the purchase price and let D be the sinking fund deposit. Now the total annual payment is split into interest on the purchase price $.09P$ and the sinking fund deposit D. Thus, we have

$$D = \frac{1000}{a_{\overline{n}|.08}} - .09P.$$

However, we know that the accumulated value of the sinking fund after n years is P. Thus, we have

$$P = Ds_{\overline{n}|.07}.$$

Thus, we have two equations in two unknowns, and solving for P

$$P = \left[\frac{1000}{a_{\overline{n}|.08}} - .09P \right] s_{\overline{n}|.07}$$

or

$$P = \frac{1000 s_{\overline{n}|.07}}{a_{\overline{n}|.08}(1 + .09 s_{\overline{n}|.07})}.$$

6.5 DIFFERING PAYMENT PERIODS AND INTEREST CONVERSION PERIODS

In Sections 6.3 and 6.4 it has been assumed that the payment period and interest conversion period are equal and coincide. Section 6.5 examines the implications of removing this assumption. We shall first analyze amortization schedules in which payments are made at a different frequency than interest is convertible.

Consider first a loan of $a_{\overline{n}|}/s_{\overline{k}|}$ which is being repaid with payments of 1 at the end of each k interest conversion periods for a total of n interest conversion periods. The number of payments is n/k, which is integral. Table 6.4 is a generalization of Table 6.1 for this situation. The reader should verify the entries in Table 6.4.

Consider next a loan of $a^{(m)}_{\overline{n}|}$ which is being repaid with payments of $1/m$ at the end of each mth of an interest conversion period for a total of n interest conversion periods. The number of payments is mn, which is integral. Table 6.5 is a generalization of Table 6.1 for this situation. The reader should verify the entries in Table 6.5.

The same observations made about the amortization schedules in Section 6.3 also apply to the amortization schedules in this section. The reader is cautioned not to rely upon the memorization of formulas in amortization and sinking fund schedules, since the reasoning used in constructing these schedules is of primary importance. Any amortization or sinking fund schedule can be constructed from basic principles.

In the case of sinking funds the approach is similar, but the situation may be a bit more complex, since the frequency of the following may differ: (1) interest payments on the loan, (2) sinking fund deposits, and (3) interest conversion period on the sinking fund.

Cases involving sinking funds with differing frequencies can be handled from basic principles. Example 6.8 is illustrative of such a case.

Table 6.4 Amortization Schedule for a Loan in Which Payments Are Made Less Frequently Than Interest Is Convertible

Period	*Payment amount*	*Interest paid*	*Principal repaid*	*Outstanding loan balance*
0				$\frac{a_{\overline{n}\rceil}}{s_{\overline{k}\rceil}}$
k	1	$[(1+i)^k - 1]\frac{a_{\overline{n}\rceil}}{s_{\overline{k}\rceil}} = 1 - v^n$	v^n	$\frac{a_{\overline{n}\rceil}}{s_{\overline{k}\rceil}} - v^n = \frac{a_{\overline{n-k}\rceil}}{s_{\overline{k}\rceil}}$
$2k$	1	$[(1+i)^k - 1]\frac{a_{\overline{n-k}\rceil}}{s_{\overline{k}\rceil}} = 1 - v^{n-k}$	v^{n-k}	$\frac{a_{\overline{n-k}\rceil}}{s_{\overline{k}\rceil}} - v^{n-k} = \frac{a_{\overline{n-2k}\rceil}}{s_{\overline{k}\rceil}}$
•	•	•	•	•
•	•	•	•	•
•	•	•	•	•
tk	1	$[(1+i)^k - 1]\frac{a_{\overline{n-(t-1)k}\rceil}}{s_{\overline{k}\rceil}} = 1 - v^{n-(t-1)k}$	$v^{n-(t-1)k}$	$\frac{a_{\overline{n-(t-1)k}\rceil}}{s_{\overline{k}\rceil}} - v^{n-(t-1)k} = \frac{a_{\overline{n-tk}\rceil}}{s_{\overline{k}\rceil}}$
•	•	•	•	•
•	•	•	•	•
•	•	•	•	•
$n-k$	1	$[(1+i)^k - 1]\frac{a_{\overline{2k}\rceil}}{s_{\overline{k}\rceil}} = 1 - v^{2k}$	v^{2k}	$\frac{a_{\overline{2k}\rceil}}{s_{\overline{k}\rceil}} - v^{2k} = \frac{a_{\overline{k}\rceil}}{s_{\overline{k}\rceil}}$
n	1	$[(1+i)^k - 1]\frac{a_{\overline{k}\rceil}}{s_{\overline{k}\rceil}} = 1 - v^k$	v^k	$\frac{a_{\overline{k}\rceil}}{s_{\overline{k}\rceil}} - v^k = 0$
Total	$\frac{n}{k}$	$\frac{n}{k} - \frac{a_{\overline{n}\rceil}}{s_{\overline{k}\rceil}}$	$\frac{a_{\overline{n}\rceil}}{s_{\overline{k}\rceil}}$	

Table 6.5 Amortization Schedule for a Loan in Which Payments Are Made More Frequently Than Interest Is Convertible

Period	*Payment amount*	*Interest paid*	*Principal repaid*	*Outstanding loan balance*
0				$a_{\overline{n}|}^{(m)}$
$\frac{1}{m}$	$\frac{1}{m}$	$\frac{i^{(m)}}{m} a_{\overline{n}|}^{(m)} = \frac{1}{m}(1 - v^n)$	$\frac{1}{m} v^n$	$a_{\overline{n}|}^{(m)} - \frac{1}{m} v^n = a_{\overline{n-\frac{1}{m}}|}^{(m)}$
$\frac{2}{m}$	$\frac{1}{m}$	$\frac{i^{(m)}}{m} a_{\overline{n-\frac{1}{m}}|}^{(m)} = \frac{1}{m}\left(1 - v^{n-\frac{1}{m}}\right)$	$\frac{1}{m} v^{n-\frac{1}{m}}$	$a_{\overline{n-\frac{1}{m}}|}^{(m)} - \frac{1}{m} v^{n-\frac{1}{m}} = a_{\overline{n-\frac{2}{m}}|}^{(m)}$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$\frac{t}{m}$	$\frac{1}{m}$	$\frac{i^{(m)}}{m} a_{\overline{n-\frac{t-1}{m}}|}^{(m)} = \frac{1}{m}\left(1 - v^{n-\frac{t-1}{m}}\right)$	$\frac{1}{m} v^{n-\frac{t-1}{m}}$	$a_{\overline{n-\frac{t-1}{m}}|}^{(m)} - \frac{1}{m} v^{n-\frac{t-1}{m}} = a_{\overline{n-\frac{t}{m}}|}^{(m)}$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$n - \frac{1}{m}$	$\frac{1}{m}$	$\frac{i^{(m)}}{m} a_{\overline{\frac{2}{m}}|}^{(m)} = \frac{1}{m}\left(1 - v^{\frac{2}{m}}\right)$	$\frac{1}{m} v^{\frac{2}{m}}$	$a_{\overline{\frac{2}{m}}|}^{(m)} - \frac{1}{m} v^{\frac{2}{m}} = a_{\overline{\frac{1}{m}}|}^{(m)}$
n	$\frac{1}{m}$	$\frac{i^{(m)}}{m} a_{\overline{\frac{1}{m}}|}^{(m)} = \frac{1}{m}\left(1 - v^{\frac{1}{m}}\right)$	$\frac{1}{m} v^{\frac{1}{m}}$	$a_{\overline{\frac{1}{m}}|}^{(m)} - \frac{1}{m} v^{\frac{1}{m}} = 0$
Total	n	$n - a_{\overline{n}|}^{(m)}$	$a_{\overline{n}|}^{(m)}$	

Example 6.8 ***A borrower takes out a loan of $2000 for two years. Construct a sinking fund schedule if the lender receives 10% effective on the loan and if the borrower replaces the amount of the loan with semiannual deposits in a sinking fund earning 8% convertible quarterly.***

In this case all three frequencies differ: (1) interest payments on the loan are made annually, (2) sinking fund deposits are made semiannually, and (3) interest on the sinking fund is convertible quarterly.

The interest payments on the loan are $200 at the end of each year. Let the sinking fund deposit be D. Then

$$D\frac{s_{\overline{8}|.02}}{s_{\overline{2}|.02}} = 2000$$

or

$$D = 2000\frac{s_{\overline{2}|.02}}{s_{\overline{8}|.02}} = 2000\frac{2.02}{8.5830} = \$470.70.$$

The sinking fund schedule is given in Table 6.6.

Table 6.6 Sinking Fund Schedule for Example 6.8

Year	*Interest paid*	*Sinking fund deposit*	*Interest earned on sinking fund*	*Amount in sinking fund*	*Net amount of loan*
0					
1/4	0	0	0	0	2000.00
1/2	0	470.70	0	470.70	1529.30
3/4	0	0	9.41	480.11	1519.89
1	200.00	470.70	9.60	960.41	1039.59
1 1/4	0	0	19.21	979.62	1020.38
1 1/2	0	470.70	19.59	1469.91	530.09
1 3/4	0	0	29.40	1499.31	500.69
2	200.00	470.70	29.99	2000.00	0

Example 6.9 ***A debt is being amortized by means of monthly payments at an annual effective rate of interest of 11%. If the amount of principal in the third payment is $1000, find the amount of principal in the 33rd payment.***

The principal repaid column in Table 6.5 is a geometric progression with common ratio $(1 + i)^{1/m}$. The interval from the third payment to the 33rd payment is $(33 - 3)/12 = 2.5$ years. Thus, the principal in the 33rd payment is

$$1000(1.11)^{2.5} = \$1298.10.$$

6.6 VARYING SERIES OF PAYMENTS

If a loan is being repaid by the amortization method, it is possible that the borrower repays the loan with installments which are not level. The case in which all the payments are level except for an irregular final payment was considered in Section 6.3. In this section we consider more general patterns of variation. We will assume that the interest conversion period and the payment period are equal and coincide.

Consider a loan L to be repaid with n periodic installments $R_1, R_2, \ldots, R_n$. Then we have

$$L = \sum_{t=1}^{n} v^t R_t. \tag{6.8}$$

Often the series of payments R_t follows some regular pattern so that the results of Section 4.6 can be used.

If it is desired to construct an amortization schedule, it can be constructed from basic principles as in Section 6.3. Alternatively, the outstanding loan balance column can be found prospectively or retrospectively as in Section 6.2, and then the interest paid and principal repaid columns can be determined.

One pattern of variation that is fairly common involves the borrower making level payments of principal. Since successive outstanding loan balances decrease, successive interest payments will also decrease. Thus, successive total payments consisting of principal and interest will decrease. Example 6.12 is an illustration of this type.

It is possible that when a loan is amortized with varying payments, the interest due in a payment is larger than the total payment. In this case, the principal repaid would be negative and the outstanding loan balance would increase instead of decrease. The increase in the outstanding loan balance arises from interest deficiencies being *capitalized* and added to the amount of the loan. This situation is often called *negative amortization*. Example 6.13 is an illustration of this type.

It is also possible to have a varying series of payments with the sinking fund method. We will assume that the interest paid to the lender is constant each period so that only the sinking fund deposits vary.

Assume that the varying payments by the borrower are $R_1, R_2, \ldots, R_n$ and that $i \neq j$. Let the amount of the loan be denoted by L. Then the sinking fund deposit for the tth period is $R_t - iL$. Since the accumulated value of the sinking fund at the end of n periods must be L, we have

$$\begin{aligned} L &= (R_1 - iL)(1+j)^{n-1} + (R_2 - iL)(1+j)^{n-2} + \cdots + (R_n - iL) \\ &= \sum_{t=1}^{n} R_t(1+j)^{n-t} - iLs_{\overline{n}|j} \end{aligned}$$

or

$$L = \frac{\sum_{t=1}^{n} R_t(1+j)^{n-t}}{1+is_{\overline{n}|j}} = \frac{\sum_{t=1}^{n} v_j^t R_t}{1+(i-j)a_{\overline{n}|j}}. \tag{6.9}$$

If $R_t = 1$, then formula (6.7) can be obtained from formula (6.9). If $i = j$, then formula (6.9) becomes

$$L = \sum_{t=1}^{n} v^t R_t$$

which is formula (6.8). Thus, the amortization method and the sinking fund method are equivalent if $i = j$.

It should be noted that we have implicitly assumed that the sinking fund deposit $R_t - iL$ is positive. If it were negative, then it would mean that the payment in that year is not even sufficient to pay the interest on the loan. We would then have a negative sinking fund deposit, i.e. a withdrawal, from the sinking fund for that year. Example 6.14 is an illustration of this type.

Example 6.10 *A borrower is repaying a loan at 5% effective with payments at the end of each year for 10 years, such that the payment the first year is \$200, the second year \$190, and so forth, until the 10th year it is \$110. (1) Find the amount of the loan. (2) Find the principal and interest in the fifth payment.*

1. The amount of the loan is

$$L = 100a_{\overline{10}|} + 10(Da)_{\overline{10}|} = 100(7.7217) + 10\frac{10 - 7.7217}{.05} = \$1227.83.$$

2. We have:

$$\begin{aligned} R_5 &= 160.00 \\ B_4^p &= 100a_{\overline{6}|} + 10(Da)_{\overline{6}|} \\ I_5 &= iB_4^p = 100(1 - v^6) + 10(6 - a_{\overline{6}|}) \\ &= 100(1 - .74622) + 10(6 - 5.0757) = \$34.62 \\ P_5 &= R_5 - I_5 = 160.00 - 34.62 = \$125.38. \end{aligned}$$

Example 6.11 *Assuming the same payment pattern as in Example 6.10, find the amount of the loan if the borrower pays 6% effective on the loan and accumulates a sinking fund to replace the amount of the loan at 5% effective.*

Using formula (6.9) the amount of the loan is

$$\frac{100a_{\overline{10}|.05} + 10(Da)_{\overline{10}|.05}}{1 + (.06 - .05)a_{\overline{10}|.05}} = \frac{1227.83}{1 + (.01)(7.7217)} = \$1139.82.$$

This answer is less than the answer to Example 6.10(1), since these terms are less favorable to the borrower.

Example 6.12 ***A borrows $20,000 from B and agrees to repay it with 20 equal annual installments of principal plus interest on the unpaid balance at 3% effective. After 10 years B sells the right to future payments to C, at a price which yields C 5% effective over the next 5 years and 4% effective over the final 5 years. Find the price which C should pay to the nearest dollar.***

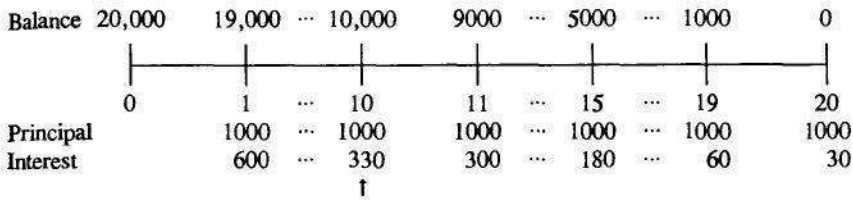

Figure 6.1 Time diagram for Example 6.12

Each year A pays $1000 principal plus interest on the unpaid balance at 3%. The payments are shown in the lower portion of Figure 6.1. The price to C at the end of the 10th year is the present value of the remaining payments, i.e.

$$1000\left[a_{\overline{5}|.05} + (1.05)^{-5}a_{\overline{5}|.04}\right] + [150a_{\overline{5}|.05} + 30(Da)_{\overline{5}|.05} + 30(1.05)^{-5}(Da)_{\overline{5}|.04}] = 1000[4.3295 + (.78353)(4.4518) + \left[150(4.3295) + 30\left[\frac{5 - 4.3295}{.05}\right] + 30(.78353)\left[\frac{5 - 4.4518}{.04}\right]\right] = \$9191$$

to the nearest dollar. The answer is less than the outstanding loan balance of $10,000, since C has a yield rate always in excess of 3%.

Example 6.13 ***A borrows $10,000 from B and agrees to repay it with a series of ten installments at the end of each year such that each installment is 20% greater than the preceding installment. The rate of interest on the loan is 10% effective. Find the amount of principal repaid in the first three installments.***

Using formula (4.39) we have

$$10{,}000 = R_1\left[\frac{1 - \left[\frac{1.2}{1.1}\right]^{10}}{.1 - .2}\right] = 13.87182\,R_1$$

or

$$R_1 = \frac{10{,}000}{13.87182} = \$720.89.$$

We now compute the entries for the first three lines of the amortization schedule. For the first line

$$I_1 = iB_0 = (.1)(10{,}000) = \$1000.00$$

and

$$P_1 = R_1 - I_1 = 720.89 - 1000.00 = -\$279.11.$$

What has happened is that the first payment is insufficient to pay interest on the loan, so the principal payment is negative. This results in an increase in the outstanding loan balance, i.e.

$$B_1 = B_0 - P_1 = 10{,}000 + 279.11 = \$10{,}279.11\,.$$

The calculations for the second line of the amortization schedule are:

$$\begin{aligned} R_2 &= 1.2R_1 = (1.2)(720.89) = \$865.07 \\ I_2 &= iB_1 = (.1)(10{,}279.11) = \$1027.91 \\ P_2 &= R_2 - I_2 = 865.07 - 1027.91 = -\$162.84 \\ B_2 &= B_1 - P_2 = 10{,}279.11 + 162.84 = \$10{,}441.95\,. \end{aligned}$$

Continuing for the third line we have:

$$\begin{aligned} R_3 &= 1.2R_2 = (1.2)(865.07) = \$1038.08 \\ I_3 &= iB_2 = (.1)(10{,}441.95) = \$1044.20 \\ P_3 &= R_3 - I_3 = 1038.08 - 1044.20 = -\$6.12 \\ B_3 &= B_2 - P_3 = 10{,}441.95 + 6.12 = \$10{,}448.07. \end{aligned}$$

Thus, the amount of principal repaid in the first three installments is

$$P_1 + P_2 + P_3 = -279.11 - 162.84 - 6.12 = -\$448.07$$

which could also be computed as $B_0 - B_3$. The principal repaid in the fourth and succeeding installments is positive and the loan is ultimately amortized to zero at the end of ten years. This example illustrates negative amortization for the first three years of the loan.

Example 6.14 What is the maximum amount that A can borrow from B if A is willing to make four successive annual payments of 100, 100, 1000 and 1000, if B is to receive 12% effective on the loan, and if A is to replace the amount of the loan in a sinking fund held by B earning 8% effective?

Let L be the amount of the loan. Annual interest on the loan is then $.12L$. The four successive sinking fund deposits are then $100 - .12L$, $100 - .12L$, $1000 - .12L$, and $1000 - .12L$. Since the accumulated value of the sinking fund must equal the original amount of the loan, we have

$$(100 - .12L)\,s_{\overline{4}|.08} + 900\,s_{\overline{2}|.08} = L$$

or

$$\begin{aligned} L &= \frac{100\,s_{\overline{4}|.08} + 900\,s_{\overline{2}|.08}}{1 + .12\,s_{\overline{4}|.08}} \\ &= \frac{(100)(4.5061) + (900)(2.08)}{1 + (.12)(4.5061)} = \$1507.47\,. \end{aligned}$$

However, this answer is subject to criticism. It is evident that the payments by A for the first two years are not sufficient to pay B 12% on the amount of the loan. Thus, the sinking fund will be negative. It is not reasonable to assume that this negative sinking fund earns 8% because that would imply that A could borrow money at 8%. Since B holds the sinking fund, it is much more reasonable to assume that A would have to borrow at 12%. Thus, the interest deficiencies during the first two years should be capitalized and added to the amount of the loan. Let L' be the amount of the loan under these conditions. We then have

$$B_2^r = L'(1.12)^2 - 100s_{\overline{2}|.12}$$

or

$$B_2^r = 1.2544L' - 212.$$

However, B_2^r is the increased amount of the loan which must be accumulated in the sinking fund. Thus, we have

$$B_2^r = \left(1000 - .12B_2^r\right)s_{\overline{2}|.08}$$

or

$$B_2^r = \frac{1000s_{\overline{2}|.08}}{1 + .12s_{\overline{2}|.08}} = \frac{1000(2.08)}{1 + (.12)(2.08)} = \$1664.53 .$$

The original loan is then calculated as

$$L' = \frac{B_2^r + 212}{1.2544} = \$1495.96.$$

Thus, L' is slightly less than L, as would be expected.

6.7 AMORTIZATION WITH CONTINUOUS PAYMENTS

It is possible to develop formulas for amortization of loans with continuous payments. Such formulas have conceptual and theoretical value, but are not widely used in practical applications.

We first examine the case in which the rate of payment is constant. Consider a loan of $\bar{a}_{\overline{n}|}$ being repaid with continuous payments at the rate of 1 per period over n periods. We can readily generalize formulas (6.1) and (6.2) to find the outstanding loan balance at time t, $0 \le t \le n$

$$B_t^p = \bar{a}_{\overline{n-t}|} \tag{6.10}$$

and

$$B_t^r = \bar{a}_{\overline{n}|}(1 + i)^t - \bar{s}_{\overline{t}|} . \tag{6.11}$$

Also, we know that the continuous payments are partially interest and partially repayment of principal. Let $\bar{I}_t$ be the instantaneous rate at which

interest is being paid and $\bar{P}_t$ be the instantaneous rate at which principal is being repaid at time t. By general reasoning, the generalizations of formulas (6.3) and (6.4) become

$$\bar{I}_t = \delta \cdot B_t \tag{6.12}$$

and

$$\bar{P}_t = 1 - \delta \cdot B_t . \tag{6.13}$$

We will verify formulas (6.12) and (6.13) if we are able to show that

$$\frac{d}{dt} B_t = - \bar{P}_t$$

since the instantaneous rate of decrease in the outstanding loan balance must equal the rate of principal repayment. If we differentiate formula (6.10), we have

$$\begin{aligned} \frac{d}{dt} B_t &= \frac{d}{dt} \bar{a}_{\overline{n-t}|} = \frac{d}{dt} \frac{1 - v^{n-t}}{\delta} \\ &= - \frac{d}{dt} \frac{(1+i)^{t-n}}{\delta} = - \frac{(1+i)^{t-n} \log_e (1+i)}{\delta} \\ &= - (1+i)^{t-n} = - v^{n-t} \\ &= \delta \bar{a}_{\overline{n-t}|} - 1 = \delta B_t - 1 = - \bar{P}_t \end{aligned}$$

from formula (6.13).

In Section 4.5 we developed the identity

$$\frac{d}{dt} \bar{a}_{\overline{t}|} = 1 - \delta \bar{a}_{\overline{t}|} \tag{4.26}$$

and indicated that it was possible to give the formula a verbal interpretation. The discussion immediately above provides that verbal interpretation.

We next consider a generalization in which the rate of payment varies. Assume that payments on the loan are being made continuously at the rate of R_t per period. This notation is consistent with the notation used in Chapter 5 from the lender's vantage point. The original amount of the loan is the present value of the payments, i.e.

$$L = B_0 = \int_0^n v^s R_s \, ds . \tag{6.14}$$

The two formulas for the outstanding loan balance then become

$$B_t^p = \int_t^n v^{s-t} R_s \, ds \tag{6.15}$$

and

$$B_t^r = B_0 (1+i)^t - \int_0^t R_s (1+i)^{t-s} ds . \tag{6.16}$$

It is interesting to note that formula (6.16) is identical to formula (5.18), which was developed in a different context, upon making the standard substitution $R_s = -C_s$.

The above formulas have assumed a constant force of interest. It is relatively straightforward to generalize the above results even further by assuming a varying force of interest. The resulting formulas would be analogous to formula (5.19) and are left as exercises.

Example 6.15 ***Show that***

$$\int_0^n v^t \bar{a}'_{\overline{n-t}|}\,dt = \int_0^n v'^t \bar{a}_{\overline{n-t}|}\,dt$$

where unprimed symbols are based on force of interest δ and primed symbols are based on force of interest δ'.

We have

$$\int_0^n v^t \bar{a}'_{\overline{n-t}|}\,dt = \int_0^n \int_0^{n-t} v^t v'^r\,dr\,dt.$$

We now reverse the order of integration as illustrated in Figure 6.2.

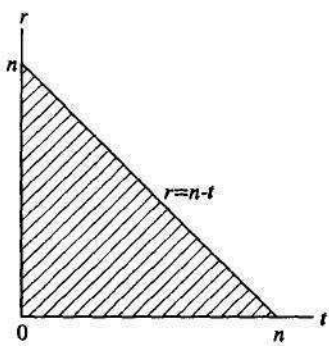

Figure 6.2 Area of integration for Example 6.15

The double integral now becomes

$$\int_0^n \int_0^{n-t} v^t v'^r\,dr\,dt = \int_0^n \int_0^{n-r} v^t v'^r\,dt\,dr$$
$$= \int_0^n v'^r \bar{a}_{\overline{n-r}|}\,dr$$

which is the result we wanted to show upon substituting t for r. The reader will probably find it surprising that this identity is true.

6.8 STEP-RATE AMOUNTS OF PRINCIPAL

A problem often encountered in practical applications involves the amortization of a loan in which the loan balance is subdivided and different rates of interest are charged on the subdivided portions. For example, a credit institution may charge 1 1/2% per month on the first \$1000 of outstanding loan balance and 1% per month on any excess. We call these subdivided portions of loan balance *step-rate amounts of principal.*

The determination of a level periodic payment which will repay a loan involving step-rate amounts of principal is surprisingly a non-trivial problem which involves trial-and-error.

In this section we give a solution to this type of problem for loans involving two step-rates. The approach can be generalized to more than two step-rates.

Consider a loan L which is being repaid with level payments of R over n periods. Interest is computed at rate i per period on the first L' of outstanding loan balance, $0 < L' < L$, and at rate j per period on the excess over L'. In practice $i > j$ in most situations, although this is not required for the derivation. It is desired to find the level payment R.

Before R can be determined it is necessary to find the "crossover" period when the outstanding loan balance first becomes less than or equal to L'. Denote this duration by a, so that a is the smallest integer such that

$$B_a \leq L'. \tag{6.17}$$

The progression of loan balances is illustrated in Figure 6.3.

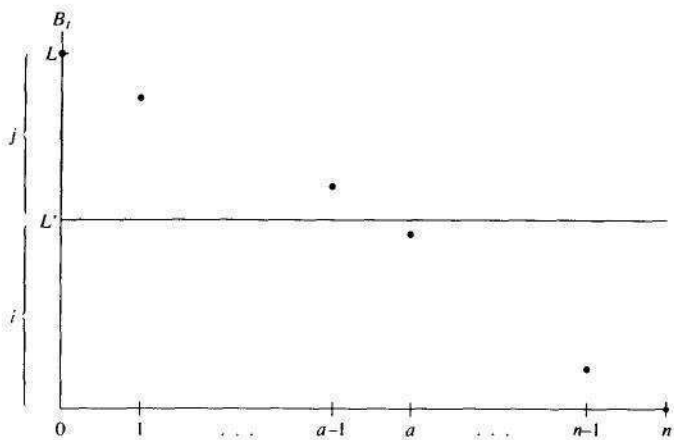

Figure 6.3 Outstanding loan balances for a loan involving step-rate amounts of principal

A formula for R can be developed by the following procedure:

1. Find B_a^p.
2. Find B_a^r.
3. Equate the two and solve for R.

The prospective approach gives

$$B_a^p = R a_{\overline{n-a}|i}. \tag{6.18}$$

The retrospective approach gives

$$B_a^r = (L - L')(1 + j)^a + L' + iL' s_{\overline{a}|j} - R s_{\overline{a}|j}. \tag{6.19}$$

Equating formulas (6.18) and (6.19) and solving for R produces

$$R = \frac{(L - L')(1 + j)^a + L' + iL' s_{\overline{a}|j}}{a_{\overline{n-a}|i} + s_{\overline{a}|j}}. \tag{6.20}$$

Formula (6.20) is the desired formula for the level periodic payment.

Although formula (6.20) is the desired formula for R, the crossover period a is unknown. A simple criterion can be developed to minimize the trial-and-error involved in finding a.

If formulas (6.18) and (6.20) are substituted into formula (6.17), then a is the smallest integer such that

$$\frac{(L - L')(1 + j)^a + L' + iL' s_{\overline{a}|j}}{a_{\overline{n-a}|i} + s_{\overline{a}|j}} a_{\overline{n-a}|i} \leq L'.$$

The above inequality is a criterion which could be used to determine a.

However, it is possible to substantially simplify the criterion as follows:

$$(L - L')(1 + j)^a a_{\overline{n-a}|i} + iL' s_{\overline{a}|j} a_{\overline{n-a}|i} \leq L' s_{\overline{a}|j}$$

$$(L - L')(1 + j)^a a_{\overline{n-a}|i} \leq L' v_i^{n-a} s_{\overline{a}|j}$$

$$\frac{L - L'}{L'} \leq \frac{a_{\overline{a}|j}}{s_{\overline{n-a}|i}}. \tag{6.21}$$

Formula (6.21) is a simple criterion which can be used to determine a.

Another computer-oriented approach to deal with this type of trial-and-error problem is to solve for R by iteration. The procedure would be as follows:

1. Determine an approximate starting value R_0. For example, R_0 could be based on a rate of interest between i and j.

2. $B_0 = L$.

3. Use the recursion formulas:

$$B_t = B_{t-1} - R + iL' + j(B_{t-1} - L'), \text{ if } B_{t-1} > L' \tag{6.22a}$$

$$B_t = B_{t-1} - R + iB_{t-1}, \text{ if } B_{t-1} \leq L'. \tag{6.22b}$$

4. $B_n = 0$.

Successive values of $R_1, R_2, \ldots$ can be generated with these formulas by standard numerical methods on a computer.

***Example 6.16** Consider a loan of \$3000 which is being repaid with level monthly payments over 12 months. Interest is computed at 1 1/2% per month on the first \$1000 of outstanding loan balance and at 1% per month on any excess over \$1000. Find the level payment which will exactly amortize this loan.*

We have $L = 3000$, $L' = 1000$, $n = 12$, $i = .015$, and $j = .01$, so that formula (6.21) gives

$$2 \leq \frac{a_{\overline{a}|.01}}{s_{\overline{12-a}|.015}}.$$

The smallest integer a satisfying this inequality is $a = 9$. The level monthly payment given by formula (6.20) is

$$R = \frac{2000(1.01)^9 + 1000 + 15s_{\overline{9}|.01}}{a_{\overline{3}|.015} + s_{\overline{9}|.01}} = 270.98545.$$

The complete amortization schedule carrying five decimal places is given in Table 6.7. The criterion is verified, since $B_8 > 1000$ and $B_9 < 1000$. The interest paid column is computed as 1 1/2% on the first \$1000 of outstanding loan balance and 1% on any excess.

Table 6.7 Amortization Schedule for a Loan of $3000 Repaid Over 12 Months at 1 1/2% Per Month on the First $1000 and 1% on Any Excess

Month	*Payment amount*	*Interest paid*	*Principal repaid*	*Oustanding loan balance*
0				3000.00000
1	270.98545	35.00000	235.98545	2764.01455
2	270.98545	32.64015	238.34530	2525.66925
3	270.98545	30.25669	240.72876	2284.94049
4	270.98545	27.84940	243.13605	2041.80445
5	270.98545	25.41804	245.56741	1796.23704
6	270.98545	22.96237	248.02308	1548.21396
7	270.98545	20.48214	250.50331	1297.71065
8	270.98545	17.97711	253.00834	1044.70231
9	270.98545	15.44702	255.53843	789.16388
10	270.98545	11.83746	259.14799	530.01589
11	270.98545	7.95024	263.03521	266.98068
12	270.98545	4.00471	266.98074	- 0.00006*

* zero, except for .00006 roundoff error

EXERCISES

6.2 Finding the outstanding loan balance

1. A loan of $1000 is being repaid with quarterly payments at the end of each quarter for five years at 6% convertible quarterly. Find the outstanding loan balance at the end of the second year.

2. A loan of $10,000 is being repaid by installments of $2000 at the end of each year, and a smaller final payment made one year after the last regular payment. Interest is at the effective rate of 12%. Find the amount of outstanding loan balance remaining when the borrower has made payments equal to the amount of the loan. Answer to the nearest dollar.

3. A loan is being repaid by quarterly installments of $1500 at the end of each quarter at 10% convertible quarterly. If the loan balance at the end of the first year is $12,000, find the original loan balance. Answer to the nearest dollar.

4. A loan is being repaid by 15 annual payments at the end of each year. The first 5 installments are $4000 each, the next 5 are $3000 each, and the final 5 are $2000 each. Find expressions for the outstanding loan balance immediately after the second $3000 installment:
   a) Prospectively.
   b) Retrospectively.

5. A loan is to be repaid with level installments payable at the end of each half-year for 3 1/2 years, at a nominal rate of interest of 8% convertible semiannually. After the fourth payment, the outstanding loan balance is \$5000. Find the initial amount of the loan. Answer to the nearest dollar.

6. A \$20,000 loan is to be repaid with annual payments at the end of each year for 12 years. If $(1 + i)^4 = 2$, find the outstanding balance immediately after the fourth payment. Answer to the nearest dollar.

7. A \$20,000 mortgage is being repaid with 20 annual installments at the end of each year. The borrower makes five payments and then is temporarily unable to make payments for the next two years. Find an expression for the revised payment to start at the end of the 8th year if the loan is still to be repaid at the end of the original 20 years.

8. A loan of 1 was originally scheduled to be repaid by 25 equal annual payments at the end of each year. An extra payment K with each of the 6th through the 10th scheduled payments will be sufficient to repay the loan 5 years earlier than under the original schedule. Show that

$$K = \frac{a_{\overline{20}|} - a_{\overline{15}|}}{a_{\overline{25}|}\, a_{\overline{5}|}}.$$

9. A loan is being repaid with level payments. If B_t, B_{t+1}, B_{t+2}, and B_{t+3} are four successive outstanding loan balances, show that:

   a) $(B_t - B_{t+1})(B_{t+2} - B_{t+3}) = (B_{t+1} - B_{t+2})^2$.

   b) $B_t + B_{t+3} < B_{t+1} + B_{t+2}$.

6.3 Amortization schedules

10. A loan is being repaid with quarterly installments of \$1000 at the end of each quarter for five years at 12% convertible quarterly. Find the amount of principal in the sixth installment.

11. Consider a loan which is being repaid with installments of 1 at the end of each period for n periods. Find an expression at issue for the present value of the interest which will be paid over the life of the loan.

12. A loan of \$10,000 is being repaid with 20 installments at the end of each year at 10% effective. Show that the amount of interest in the 11th installment is

$$\frac{1000}{1 + v^{10}}.$$

13. A loan is being repaid with 20 installments at the end of each year at 9% effective. In what installment are the principal and interest portions most nearly equal to each other?

14. A loan is being repaid with a series of payments at the end of each quarter for five years. If the amount of principal in the third payment is \$100, find the amount of principal in the last five payments. Interest is at the rate of 10% convertible quarterly.

15. A loan is being repaid with installments of 1 at the end of each year for 20 years. Interest is at effective rate i for the first 10 years and effective rate j for the second 10 years. Find expressions for:
    a) The amount of interest paid in the 5th installment.
    b) The amount of principal repaid in the 15th installment.

16. A mortgage with original principal A is being repaid with level payments of K at the end of each year for as long as necessary plus a smaller final payment. The effective rate of interest is i.
    a) Find the amount of principal in the tth installment.
    b) Is the principal repaid column in the amortization schedule in geometric progression (excluding the irregular final payment)?

17. A borrower has a mortgage which calls for level annual payments of 1 at the end of each year for 20 years. At the time of the seventh regular payment an additional payment is made equal to the amount of principal that according to the original amortization schedule would have been repaid by the eighth regular payment. If payments of 1 continue to be made at the end of the eighth and succeeding years until the mortgage is fully repaid, show that the amount saved in interest payments over the full term of the mortgage is

$$1 - v^{13}.$$

18. A loan of L is being amortized with payments at the end of each year for 10 years. If $v^5 = 2/3$, find the following:
    a) The amount of principal repaid in the first 5 payments.
    b) The amount due at the end of 10 years if the final 5 payments are not made as scheduled.

19. A 35-year loan is to be repaid with equal installments at the end of each year. The amount of interest paid in the 8th installment is \$135. The amount of interest paid in the 22nd installment is \$108. Calculate the amount of interest paid in the 29th installment.

6.4 Sinking funds

20. A has borrowed \$10,000 on which interest is charged at 10% effective. A is accumulating a sinking fund at 8% effective to repay the loan. At the end of 10 years the balance in the sinking fund is \$5000. At the end of the 11th year A makes a total payment of \$1500.
    a) How much of the \$1500 pays interest currently on the loan?
    b) How much of the \$1500 goes into the sinking fund?
    c) How much of the \$1500 should be considered as interest?

d) How much of the \$1500 should be considered as principal?
e) What is the sinking fund balance at the end of the 11th year?

21. A loan of \$1000 is being repaid with level annual payments of \$120 plus a smaller final payment made one year after the last regular payment. The effective rate of interest is 8%. Show algebraically and verbally that the outstanding loan balance after the fifth payment has been made is:
a) $1000(1.08)^5 - 120s_{\overline{5}|}$.
b) $1000 - 40s_{\overline{5}|}$.

22. Show that $a_{\overline{n}|i\&j} = \dfrac{s_{\overline{n}|j}}{1 + is_{\overline{n}|j}}$.

23. On a loan of \$10,000 interest at 9% effective must be paid at the end of each year. The borrower also deposits \$$X$ at the beginning of each year into a sinking fund earning 7% effective. At the end of 10 years the sinking fund is exactly sufficient to pay off the loan. Calculate X.

24. A borrower is repaying a loan with 10 annual payments of \$1000. Half of the loan is repaid by the amortization method at 5% effective. The other half of the loan is repaid by the sinking fund method in which the lender receives 5% effective on the investment and the sinking fund accumulates at 4% effective. Find the amount of the loan. Answer to the nearest dollar.

25. A borrows \$12,000 for 10 years and agrees to make semiannual payments of \$1000. The lender receives 12% convertible semiannually on the investment each year for the first 5 years and 10% convertible semiannually for the second 5 years. The balance of each payment is invested in a sinking fund earning 8% convertible semiannually. Find the amount by which the sinking fund is short of repaying the loan at the end of the 10 years. Answer to the nearest dollar.

26. *a*) A borrower takes out a loan of \$3000 for 10 years at 8% convertible semiannually. The borrower replaces one third of the principal in a sinking fund earning 5% convertible semiannually and the other two thirds in a sinking fund earning 7% convertible semiannually. Find the total semiannual payment.
b) Rework (*a*) if the borrower each year puts one third of the total sinking fund deposit into the 5% sinking fund and the other two thirds into the 7% sinking fund.
c) Justify from general reasoning the relative magnitude of the answers to (*a*) and (*b*).

27. A payment of \$36,000 is made at the end of each year for 31 years to repay a loan of \$400,000. If the borrower replaces the capital by means of a sinking fund earning 3% effective, find the effective rate paid to the lender on the loan.

28. A 20-year annuity-immediate has a present value of \$10,000 where interest is 8% effective for the first 10 years and 7% effective for the second 10 years. An investor buys this annuity at a price which over the entire period yields 9% on the purchase price and further allows the replacement of capital by means of a sinking

fund earning 6% for the first 10 years and 5% for the second 10 years. Find an expression for the amount that is placed in the sinking fund each year.

29. A loan of 1 yields the lender rate i per period for n periods, while the borrower replaces the capital in a sinking fund earning rate j per period. Find expressions for the following if $1 \leq t \leq n$:
    a) Periodic interest paid to the lender.
    b) Periodic sinking fund deposit.
    c) Interest earned on sinking fund during period t.
    d) Amount in sinking fund at end of t periods.
    e) Net amount of loan at the end of t periods.
    f) Net interest paid in period t.
    g) Principal repaid in period t.

6.5 Differing payment periods and interest conversion periods

30. An investor buys an annuity with payments of principal and interest of \$500 per quarter for 10 years. Interest is at the effective rate of 8% per annum. How much interest does the investor receive in total over the 10-year period? Answer to the nearest dollar.

31. A borrows \$10,000 for five years at 12% convertible semiannually. A replaces the principal by means of deposits at the end of every year for five years into a sinking fund which earns 8% effective. Find the total dollar amount which A must pay over the five-year period to completely repay the loan. Answer to the nearest dollar.

32. A borrower is repaying a loan with payments of \$3000 at the end of every year over an unknown period of time. If the amount of interest in the third installment is \$2000, find the amount of principal in the sixth installment. Assume that interest is 10% convertible quarterly.

33. A borrows \$5000 for 10 years at 10% convertible quarterly. A does not pay interest currently and will pay all accrued interest at the end of 10 years together with the principal. Find the annual sinking fund deposit necessary to liquidate the loan at the end of 10 years if the sinking fund earns 7% convertible semiannually.

34. A loan of \$3000 is being amortized by 20 quarterly payments. Payments 11 and 12 are not made. At the designated time of the 12th payment, the loan is renegotiated so that the 13th payment is \$$N$ and payments 14, 16, 18, and 20 are each \$40 more than the preceding payment. If the rate of interest is 8% convertible semiannually, find the value of N which would provide that the loan be completely repaid in the original time period. Answer to the nearest dollar.

6.6 Varying series of payments

35. A loan is being repaid with 10 annual payments. The first payment is equal to the interest due only, the second payment is twice the first, the third payment is three times the first, and so forth. Prove that at the rate of interest on the loan

$$(Ia)_{\overline{10}|} = a_{\overline{\infty}|} .$$

36. A loan is being repaid with 10 payments. The first payment is 10, the second 9, and so forth with the tenth payment being 1. Show that the amount of interest in the sixth payment is

$$5 - a_{\overline{5}|} .$$

37. A loan is repaid with payments which start at \$200 the first year and increase by \$50 per year until a payment of \$1000 is made, at which time payments cease. If interest is 4% effective, find the amount of principal in the fourth payment.

38. A borrower is repaying a \$1000 loan with 10 equal payments of principal. Interest at 6% convertible semiannually is paid on the outstanding balance each year. Find the price to yield an investor 10% convertible semiannually.

39. A mortgage of \$8000 is repayable in 20 years by semiannual installments of \$200 each plus interest on the unpaid balance at 5%. Just after the 15th payment the lender sells the mortgage at a price which yields the new lender 6% and allows the accumulation of a sinking fund to replace the capital at 4%. Assume that all interest rates are convertible semiannually.

a) Show that the price assuming a level net return every six months is

$$\frac{75 s_{\overline{25}|\ .02} + 6250}{1 + .03 s_{\overline{25}|\ .02}} = \$4412.$$

b) Show that the price assuming a level sinking fund deposit every six months is

$$\frac{(a_{\overline{25}|\ .03} + 125) s_{\overline{25}|\ .02}}{.03 a_{\overline{25}|\ .03} (1 + .03 s_{\overline{25}|\ .02})} = \$4453.$$

c) Justify from general reasoning the relative magnitude of the answers to (*a*) and (*b*).

40. A borrows \$2000 at an effective rate of interest of 10% per annum and agrees to repay the loan with payments at the end of each year. The first payment is to be \$400 and each payment thereafter is to be 4% greater than the preceding payment, with a smaller final payment made one year after the last regular payment.
a) Find the outstanding loan balance at the end of three years.
b) Find the principal repaid in the third payment.

41. Two loans for equal amounts are amortized at 4% effective. Loan L is to be repaid by 30 equal annual payments. Loan M is to be repaid by 30 annual payments, each containing equal principal amounts with the interest portion of each payment based upon the unpaid balance. The payment for loan L first exceeds the payment for loan M at the end of year k. Find k.

42. A has money invested at effective rate i. At the end of the first year A withdraws 162 1/2% of the interest earned, at the end of the second year A withdraws 325%

of the interest earned, and so forth with the withdrawal factor increasing in arithmetic progression. At the end of 16 years the fund is exhausted. Find i.

6.7 Amortization with continuous payments

43. A loan of $\bar{a}_{\overline{25}|}$ is being repaid with continuous payments at the annual rate of 1 per annum for 25 years. If $i = .05$, find the total amount of interest paid during the sixth through the tenth years inclusive.

44. *a*) Show that
$$(1+i)^t - \frac{\bar{s}_{\overline{t}|}}{\bar{a}_{\overline{n}|}} = \frac{\bar{a}_{\overline{n-t}|}}{\bar{a}_{\overline{n}|}}.$$

b) Verbally interpret the result obtained in (*a*).

45. A loan is being repaid over n periods with continuous payments at the rate of t per period at time t. Find expressions for the outstanding loan balance at time $k, 0 \le k \le n$:
   a) Prospectively.
   b) Retrospectively.

46. A loan of 1 is being amortized over a 10-year period with continuous payments which vary in such a fashion that the outstanding loan balance is linear. The force of interest is 10%. Find:
   a) The principal repaid over the first 5 years.
   b) The interest paid over the first 5 years.

47. Express formulas (6.14), (6.15), and (6.16) in more general form in which the force of interest varies continuously.

48. It is known that the remaining undiscounted payout on an insurance claim t periods after the claim was incurred is given by $\alpha e^{-\beta t}$.
   a) If the instantaneous rate of claim payment is $P(t)$, find an expression for $P(t)$.
   b) Find the undiscounted total payout on the claim at time 0.
   c) Find the present value of the total payout on the claim at time 0, if the force of interest is δ.
   d) Find the present value of the remaining payout on the claim at time t, if the force of interest is δ.

6.8 Step-rate amounts of principal

49. In order to pay off a \$2000 loan, payments of \$$P$ are made at the end of each quarter. Interest on the first \$500 of the unpaid balance is at rate $i^{(4)} = 16\%$, while interest on the excess is at $i^{(4)} = 14\%$. If the outstanding loan balance is \$1000 at the end of the first year, find P. Answer to the nearest dollar.

50. A loan of \$1000 is to be amortized with quarterly installments of \$100 for as long as necessary plus a smaller final payment one quarter after the last regular payment.

Interest is computed at 12% convertible quarterly on the first \$500 of outstanding loan balance and at 8% convertible quarterly on any excess.

a) Find the principal repaid in the fourth installment.

b) Show that prior to the "crossover" point, the successive principal repayments plus a constant form a geometric progression, i.e.

$$\frac{P_{t+1} + k}{P_t + k} = 1 + j \text{ for } t = 1, 2, \ldots, a - 1.$$

(1) Find k.

(2) Find j.

51. Consider a loan of \$3000 which is being repaid with level monthly payments over 12 months. Interest is computed at 1 1/2% per month on the first \$1000 of outstanding loan balance, at 1 1/4% per month on the next \$1000, and at 1% per month on any excess over \$1000. Find the level payment which will exactly amortize this loan. (Hint: Assume that the two "crossover" points are $t = 5$ and $t = 9$. These can be confirmed as correct from the resulting amortization schedule.)

Miscellaneous problems

52. *a*) Show that

$$a_{\overline{n}|} + i \sum_{t=0}^{n-1} a_{\overline{n-t}|} = n.$$

b) Verbally interpret the result obtained in (*a*).

53. *a*) Show that if a loan is amortized with n level payments of R

$$B_t = R(a_{\overline{n}|} - v^n s_{\overline{t}|}).$$

b) Verbally interpret the result obtained in (*a*).

54. The original amount of an inheritance was just sufficient at 3 1/2% effective to pay \$10,000 at the end of each year for 10 years. The payments were made as scheduled for the first five years even though the fund actually earned 5% effective. How much excess interest was in the fund at the end of the fifth year? Answer to the nearest dollar.

55. An investor is making level payments at the beginning of each year for 10 years to accumulate \$10,000 at the end of the 10 years in a bank which is paying 5% effective. At the end of five years the bank drops its interest rate to 4% effective.

a) Find the annual deposit for the first five years.

b) Find the annual deposit for the second five years.

56. A family was making annual payments of R on a 10% 30-year mortgage. After making 15 payments they renegotiate to pay off the debt in 5 more years with the lender being satisfied with 9% effective over the entire period. Find an expression for the revised annual payment.

57. Nine years ago a family incurred a 20-year \$80,000 mortgage at 8% effective on which they were making annual payments. They desire now to make a lump-sum payment of \$5000 and to pay off the mortgage in nine more years. Find an expression for the revised annual payment:
    a) If the lender is satisfied with an 8% yield for the past nine years but insists on a 9% yield for the next nine years.
    b) If the lender insists on a 9% yield during the entire life of the mortgage.

58. A loan of \$1000 is to be repaid with equal annual payments at 5% effective over a 10-year period. The borrower may accelerate the amortization of the loan. However, there is a prepayment penalty of 2% of the excess of any payment over the originally scheduled payment. If the borrower makes a \$300 payment at the end of the first year and a \$250 payment at the end of the second year, find the outstanding loan balance just prior to the payment at the end of the third year. Answer to nearest dollar.

7

Bonds and other securities

7.1 INTRODUCTION

One of the major applications of the theory of interest is in the determination of prices and values for bonds and other securities, such as preferred stock and common stock. This chapter is mainly concerned with bonds, although brief consideration is given to preferred and common stock in Sections 7.2, 7.10, and 7.11.

There are three main questions which Chapter 7 considers:

1. Given the desired yield rate of an investor, what price should be paid for a given security?
2. Given the purchase price of a security, what is the resulting yield rate to an investor?
3. What is the value of a security on a given date after it has been purchased?

It should be noted that no new basic principles are introduced in this chapter. However, several new terms and formulas are introduced to efficiently handle financial calculations involving these types of securities.

7.2 TYPES OF SECURITIES

Section 7.2 considers three common types of securities which have evolved in the financial markets. This section should not be considered as a complete discussion of these securities, since only a very brief description is presented. For a more complete discussion the reader is encouraged to refer to any of several standard textbooks on finance or securities.

In recent years there has been a proliferation of more exotic financial instruments. These are not discussed at this stage, since it is important to deal

with the basics first. However, these newer financial instruments are discussed in Section 8.8.

The three main categories of securities which will be discussed in this chapter are: (1) bonds, (2) preferred stock, and (3) common stock.

Bonds

A *bond* is an interest-bearing security which promises to pay a stated amount (or amounts) of money at some future date (or dates). It is a formal certificate of indebtedness issued by a borrower, usually for some round figure such as $1000 or $5000. Bonds are commonly issued by corporations and governmental units as a means of raising capital.

Bonds are generally redeemed at the end of a fixed period of time. This fixed period of time is called the *term* of the bond. The end of the term of a bond is called the *maturity date*. On occasion, bonds with an infinite term are issued, e.g. the British consols. Such bonds are called *perpetuals*. Also bonds may be issued with a term which varies at the discretion of the borrower. Such a bond is termed a *callable bond* and is discussed in Section 7.7. Any date prior to, or including, the maturity date on which a bond may be redeemed is termed a *redemption date*.

Bonds may be classified in several different ways. One such classification is the distinction between *accumulation bonds* and *bonds with coupons*. The coupons are periodic payments made by the issuer of the bond prior to its redemption. An accumulation bond is one in which the redemption price includes the original loan plus all accumulated interest. The Series E Savings Bonds issued by the United States Treasury are a traditional example of this type of bond. More recently, so-called *zero coupon bonds* have become very popular with investors. However, most bonds have coupons payable periodically, and this will be assumed unless stated otherwise. This chapter will largely be confined to this latter type of bond, since accumulation or zero coupon bonds can easily be handled with elementary compound interest methods already discussed in earlier chapters.

A second classification is the distinction between *registered bonds* and *unregistered bonds*. A registered bond is one in which the lender is listed in the records of the borrower. If the lender decides to sell the bond, the change of ownership must be reported to the borrower. The periodic coupon payments are paid by the borrower to the owners of record on each coupon payment date. An unregistered bond is one in which the lender is not listed in the records of the borrower. Since the bond belongs to whomever has legal possession of it, an unregistered bond is often called a *bearer bond*. Unregistered bonds are almost always issued with coupons attached. The coupons can be detached by

the holder of the bond and cashed. Such a bond is often called a *coupon bond* for this reason.

A third classification is made according to the type of security behind the bond. A *mortgage bond* is a bond secured by a mortgage on real property. A *debenture bond* is one secured only by the general credit of the borrower. Variations within these two major classifications exist. In general, mortgage bonds possess a higher degree of security than debenture bonds, since the lenders can foreclose on the collateral in the event of the default of a mortgage bond.

A type of bond with a high degree of risk to the lender is the *income bond* or *adjustment bond*. Under this type of bond, the periodic coupons are paid only if the borrower has earned sufficient income to pay them. Income bonds were once fairly common, but they have largely disappeared in recent years.

The more modern version of a *high-risk bond* is often called a *"junk" bond*. Such bonds are issued by corporations, typically in connection with corporate mergers, acquisitions, or other takeovers. They have a significantly higher risk of default in payments than corporate bonds in general. Accordingly, they must pay commensurately higher rates of interest to the lender for assuming this risk. Such bonds are sometimes characterized as *below investment grade*. The phrase "investment grade" refers to a schedule of quality rankings for various bonds provided by certain rating organizations. Financial calculations involving the risk of default will not be covered in this chapter, but will be addressed in Section 9.5.

A type of bond which is somewhat of a hybrid is a *convertible bond*. This type of bond can be converted into the common stock of the issuing corporation at some future date under certain conditions, at the option of the owner of the bond. Convertible bonds are generally debenture bonds. Such bonds offer an investor a choice between continuing the security as a bond or switching it into common stock, depending upon the performance of the corporation in the future.

A borrower in need of a large amount of funds can issue bonds with a common maturity date. However, a large volume of indebtedness falling due at one time can present problems in redeeming or refinancing the debt. For this reason, some borrowers will divide a large issue of bonds so that individual bonds will have a series of staggered redemption dates. These types of bonds are called *serial bonds* and will be analyzed further in Section 7.8. An alternative approach is for the lenders to require the establishment of a sinking fund to build up the amount necessary to repay the indebtedness for a large issue of bonds.

The United States Treasury issues indebtedness with a wide range of maturities. Long-term debt of seven or more years duration is issued in the form of *Treasury bonds*. Short-term debt is issued in the form of *Treasury bills*, often called "T-bills." These are issued on a discount basis for maturities of 13, 26, or 52 weeks, and were discussed briefly in Section 2.8. Debt of an intermediate duration, i.e. one to seven years, is typically issued in the form of *Treasury notes*. Although called "notes," these securities are mathematically equivalent to bonds; the term is just shorter. Calculations involving Treasury bills have some significant differences from those on longer-term Treasury securities. These differences are explored in Example 7.2.

Preferred stock

Preferred stock is a type of security which provides a fixed rate of return similar to bonds. However, it differs from a bond in that it is an ownership security rather than a debt security, i.e. the owner of preferred stock is part owner of the issuing corporation, while the bond owner is a creditor of the corporation. In general, preferred stock has no maturity date, although on occasion preferred stock with a redemption provision is issued. The periodic payment on preferred stock is usually called a *dividend*, since it is being paid to an owner.

In terms of the degree of security, preferred stock ranks behind bonds and other debt instruments, since all payments on indebtedness must be made before the preferred stock receives a dividend. However, preferred stock ranks ahead of common stock in the degree of security involved, since preferred stock dividends must be paid before common stock dividends can be paid.

To increase the degree of security behind preferred stock, some corporations have issued *cumulative preferred stock*. This type of preferred stock has the feature that any dividends which the corporation is not able to pay are carried forward to future years when they presumably will be paid. For example, if a corporation has preferred stock on which it is paying a $5 dividend per share and can make only a $3 payment in one year, the $2 balance is carried forward (perhaps without interest) to future years. All arrears on preferred stock must be paid before any dividends on common stock can be paid.

Some preferred stocks receive a share of earnings over and above the regular dividend if earnings are at a sufficient level. This type of preferred stock is called *participating preferred stock*, since it participates in the earnings with the common stock. Participating preferred stock is relatively uncommon at the present time.

Some preferred stock has a convertible privilege similar to convertible bonds and is called *convertible preferred stock*. Owners of this type of preferred stock

have the option to convert their preferred stock to common stock under certain conditions.

Common stock

Common stock is a type of ownership security, as is preferred stock. However, it does not earn a fixed dividend rate as preferred stock does. Common stock dividends are paid only after interest payments on all bonds and other debt and dividends on preferred stock are paid. The dividend rate is completely flexible and can be set by the corporation's board of directors at its discretion.

Since the dividend rates on common stocks are variable, the prices of common stocks tend to be much more volatile than either bonds or preferred stocks. However, all residual profits after dividends to the preferred stockholders belong to the common stockholders.

Example 7.1 A zero coupon bond will pay $1000 at the end of 10 years and is currently selling for $400. Find the yield rate convertible semiannually that would be earned by a purchaser.

Let j be the yield rate per half-year. The equation of value is

$$400(1+j)^{20} = 1000$$

or

$$(1+j)^{20} = 2.5$$

which gives $j = .0469$. Thus, the yield rate convertible semiannually is equal to $2(.0469) = .0938$, or 9.38%.

Example 7.2 A 13-week Treasury bill matures for $10,000 and is bought at discount to yield 7.5%. Find the price which should be paid.

As noted in Section 2.8, T-bill yields are computed as rates of discount rather than rates of interest. These yields are computed on a simple discount basis, which, in effect, results in a rate of discount convertible at the same frequency as the term of the T-bill. Also, it is necessary to know the basis for measuring time periods as described in Section 2.3. The basis typically used for T-bills is actual/360. On this basis the price would be

$$10{,}000\left[1-\frac{91}{360}(.075)\right] = \$9810.42.$$

It should be noted that the basis for determining time periods for longer-term Treasury securities is actual/actual rather than actual/360.

7.3 PRICE OF A BOND

As mentioned in Section 7.1, one of the three primary questions under consideration in this chapter is the determination of a purchase price which will

produce a given yield rate to an investor. This section considers this question for bonds.

The following assumptions are made:

1. All obligations will be paid by the bond issuer on the specified dates of payment. Market prices of bonds will vary depending upon the probability of default in payments, but we will ignore any possibility of default in this chapter. Financial calculations reflecting the probability of default will be discussed in Section 9.5.
2. The bond has a fixed maturity date. Bonds with no maturity date are mathematically equivalent to preferred stock and are considered in Section 7.10. Callable bonds with optional redemption dates are considered in Section 7.7.
3. The price of the bond is desired immediately after an coupon payment date. The price of a bond between two coupon payment dates is considered in Section 7.5.

The following symbols will be used in connection with bonds in this and succeeding sections:

P = the *price* of a bond.

F = the *par value* or *face amount* of a bond. This value is printed on the front of the bond and is often the amount payable at the maturity date. Its sole purpose is to define the series of payments to be made by the borrower. It is not a measure of the price or the value of a bond prior to maturity. It is customary to quote bond prices in terms of a par value of \$100, even though bonds are rarely issued in such small denominations, and this will be assumed unless stated otherwise.

C = the *redemption value* of a bond, i.e. the amount of money paid at a redemption date to the holder of the bond. Often C is equal to F; for example, the common case of a bond redeemed at its maturity date for its par value. It is possible for C to differ from F in the following cases: (1) a bond which matures for an amount not equal to its par value, or (2) a bond which has a redemption date prior to the maturity date on which the bond is redeemed for an amount not equal to its par value. It will be assumed that a bond is redeemable at par unless stated otherwise. The reader should note that the symbol C was also used in Section 5.5 to represent the total net amount of principal contributed in determining the yield rate on an investment fund.

r = the *coupon rate* of a bond, i.e. the rate per coupon payment period used in determining the amount of the coupon. The most common frequency

for bond coupons in the United States is semiannual. For example, an 8% bond with semiannual coupons has $r = .04$. In international financial markets other coupon frequencies are commonly encountered, e.g. Eurobonds generally have annual coupons. It is assumed that coupons are constant. The case of varying coupons is considered in Section 7.9. The reader should note that the symbol r was also used in Section 5.9 to represent the project return rate.

Fr = the amount of the coupon.

g = the *modified coupon rate* of a bond. The rate g is defined by $Fr = Cg$ or $g = Fr/C$, i.e. g is the coupon rate per unit of redemption value rather than per unit of par value. It should be noted that g will always be convertible at the same frequency as r. In practice, g will often equal r, which is the case whenever C is equal to F.

i = the *yield rate* of a bond, often called the *yield to maturity*, i.e. the interest rate actually earned by the investor, assuming the bond is held until it is redeemed or matures. The yield to maturity is identical in concept to the internal rate of return as defined in Section 5.2. It is customary that yield rates are convertible at the same frequency as the coupon rate, and this will be assumed unless stated otherwise. The case in which the coupon payment period and the yield rate conversion period are not equal is considered in Section 7.9. It is also assumed that the yield rate is constant. The case in which this is not true is also discussed in Section 7.9.

n = the number of coupon payment periods from the date of calculation to the maturity date, or to a redemption date.

K = the present value, computed at the yield rate, of the redemption value at the maturity date, or a redemption date, i.e. $K = Cv^n$ at the yield rate i.

G = the *base amount* of a bond. The amount G is defined by $Gi = Fr$ or $G = Fr/i$. Thus, G is the amount which, if invested at the yield rate i, would produce periodic interest payments equal to the coupons on the bond.

The reader should be aware that in everyday business and financial usage there are three different "yields" associated with a bond:

1. *Nominal yield* is simply the annualized coupon rate on the bond. For example, if a $100 par value bond has coupons totalling $9 per year, then the nominal yield on the bond is 9% per annum. The reader should note that the use of the word "nominal" in this context is different than the

meaning in Section 1.8 and yet another meaning to be introduced in Section 9.4, which is an unfortunate source of ambiguity.

2. *Current yield* is the ratio of the annualized coupon to the original price of the bond. For example, if the bond described above is selling for \$90 in the market, then the current yield on the bond is 10% per annum. Note that the current yield does not reflect any gain or loss when the bond is sold, is redeemed, or matures.

3. *Yield to maturity* is the actual annualized yield rate, or internal rate of return, as defined in Section 5.2. The determination of the yield to maturity will be discussed in Section 7.6.

We will use the phrase "yield rate" to refer only to the third of these meanings. However, again the reader is reminded that in practical applications not everyone uses terms with textbook precision, and it is important to ascertain exactly what is meant by the user when encountering terms such as the "yield" on a bond.

The reader should be aware that F, C, r, g, and n are given by the terms of a bond and remain fixed throughout its life. In essence, these parameters define exactly what payments will be made by the borrower. On the other hand, P and i will vary throughout the life of the bond. The price and the yield rate have a precise inverse relationship to each other, i.e. as the price rises the yield rate falls, and vice versa. Yield rates on bonds will fluctuate with prevailing rates of interest in financial markets for securities of a similar type. Fluctuating market rates of interest will thus lead to fluctuating bond prices. However, these fluctuating bond prices do not generally reflect any increase or decrease in the degree of safety or security attributed to the bond, but rather they merely reflect changing rates of interest in the securities markets. This inverse relationship between bond prices and yield rates has not always been understood by unsophisticated bondholders, who, in periods of rising rates of interest, have attributed the declining prices of their bonds to a deterioration in the credit rating of the borrower, when this was not a factor at all.

There are four types of formulas which can be used to find the price of a bond. The first of these, called the *basic formula*, is the most straightforward. According to this method the price must be equal to the present value of future coupons plus the present value of the redemption value

$$P = Fr\,a_{\overline{n}|} + Cv^n = Fr\,a_{\overline{n}|} + K \tag{7.1}$$

where the interest functions are calculated at the yield rate i.

The second formula, called the *premium/discount formula*, can be obtained from formula (7.1)

$$\begin{aligned} P &= Fr\,a_{\overline{n}|} + Cv^n \\ &= Fr\,a_{\overline{n}|} + C(1 - i a_{\overline{n}|}) \\ &= C + (Fr - Ci)\,a_{\overline{n}|}. \end{aligned} \tag{7.2}$$

The rationale for the name of this method will become evident in Section 7.4. Formula (7.2) is often more efficient than formula (7.1), since only one value need be computed or looked up in the interest tables instead of two.

The third formula, called the *base amount formula*, can also be obtained from formula (7.1)

$$\begin{aligned} P &= Fr\,a_{\overline{n}|} + Cv^n \\ &= Gi\,a_{\overline{n}|} + Cv^n \\ &= G(1 - v^n) + Cv^n \\ &= G + (C - G)v^n. \end{aligned} \tag{7.3}$$

Formula (7.3) also requires only one value instead of two.

The fourth formula, called the *Makeham formula* (after a famous British actuary of the 19th century), can also be obtained from formula (7.1)

$$\begin{aligned} P &= Cv^n + Fr\,a_{\overline{n}|} \\ &= Cv^n + Cg\left[\frac{1 - v^n}{i}\right] \\ &= Cv^n + \frac{g}{i}(C - Cv^n) \\ &= K + \frac{g}{i}(C - K). \end{aligned} \tag{7.4}$$

Formula (7.4) also requires only one value instead of two.

Makeham's formula has an interesting verbal interpretation. From the basic formula, the price of a bond must be equal to the present value of the redemption value plus the present value of future coupons. K is present value of the redemption value. If the modified coupon rate g is equal to the yield rate i, then the bond will sell for its redemption value C, so that $C - K$ is the present value of future coupons. However, if the modified coupon rate g is different than the yield rate i, then the present value of future coupons will be proportionally higher or lower, i.e. it will be equal to $\frac{g}{i}(C - K)$.

Example 7.3 Find the price of a \$1000 par value 10-year bond with coupons at 8.4% convertible semiannually, which will be redeemed at \$1050.

The bond is bought to yield 10% convertible semiannually. Use all four formulas.

In this example we have:

$$\begin{aligned} F &= 1000 \\ C &= 1050 \\ r &= \frac{.084}{2} = .042 \\ g &= \frac{1000}{1050}(.042) = .04 \\ i &= \frac{.10}{2} = .05 \\ n &= 20 \\ K &= 1050(1.05)^{-20} = 395.7340 \\ G &= \frac{.042}{.05}(1000) = 840. \end{aligned}$$

1. *Basic formula*:

$$\begin{aligned} P &= Fr\,a_{\overline{n}|} + K \\ &= 42a_{\overline{20}|.05} + 395.7340 \\ &= (42)(12.4622) + 395.7340 = \$919.15. \end{aligned}$$

2. *Premium/discount formula*:

$$\begin{aligned} P &= C + (Fr - Ci)a_{\overline{n}|} \\ &= 1050 + (42 - 52.50)a_{\overline{20}|.05} \\ &= 1050 + (42 - 52.50)(12.4622) = \$919.15. \end{aligned}$$

3. *Base amount formula*:

$$\begin{aligned} P &= G + (C - G)v^n \\ &= 840 + (1050 - 840)(1.05)^{-20} \\ &= 840 + (1050 - 840)(.37689) = \$919.15. \end{aligned}$$

4. *Makeham formula*:

$$P = K + \frac{g}{i}(C - K)$$

$$= 395.7340 + \frac{.04}{.05}(1050 - 395.7340) = \$919.15.$$

7.4 PREMIUM AND DISCOUNT

If the purchase price of a bond exceeds its redemption value, i.e. if $P > C$, then the bond is said to sell at a *premium* and the difference between P and C is called the "premium." Similarly, if the purchase price is less than the redemption value, i.e. if $P < C$, then the bond is said to sell at a *discount*, and the difference between C and P is called the "discount." The reader should note that this is yet another meaning for the overused word "discount" in addition to the various meanings given in Chapter 1.

We can derive expressions for the premium and the discount from formula (7.2):

$$\text{Premium} = P - C = (Fr - Ci)a_{\overline{n}|i} = C(g - i)a_{\overline{n}|i} \text{ if } g > i \quad (7.5)$$

$$\text{Discount} = C - P = (Ci - Fr)a_{\overline{n}|i} = C(i - g)a_{\overline{n}|i} \text{ if } i > g \quad (7.6)$$

It is clear that premium and discount, although bearing different names, are essentially the same concept, since discount is merely a negative premium. It should also be noted that in many cases $F = C$ and therefore $g = r$. However, formulas (7.5) and (7.6) will handle the cases in which this is not true.

The price of a bond depends upon two quantities, the present value of future coupons and the present value of the redemption value. Since the purchase price of a bond is usually less than or greater than the redemption value, there will be a profit (equal to the discount) or a loss (equal to the premium) at the redemption date. This profit or loss is reflected in the yield rate for the bond, when calculated as the yield to maturity.

However, as a result of this profit or loss at the redemption date, the amount of each coupon cannot be considered as interest income to an investor. It will be necessary to divide each coupon into interest earned and principal adjustment portions similar to the separation of payments into interest and principal in Chapter 6.

When this approach is used, the value of the bond will be continually adjusted from the price on the purchase date to the redemption value on the redemption date. These adjusted values of the bond are called the *book values* of the bond. The book values provide a reasonable and smooth series of values for bonds and are used by many investors, such as insurance companies and pension funds, in reporting the asset values of bonds for financial statements. The determination of asset values for bonds and other securities will be discussed further in Section 7.11.

The reader should be careful to note, however, that the book value after purchase will differ from the price of the bond if the bond were bought anew. The price of the bond in the market will vary as prevailing interest rates change. However, the book values will follow a smooth progression, since they are all based on the original yield rate at purchase.

When convenient, we will use the same notation for bonds as developed in Chapter 6 for loans. Thus, the book value t periods after purchase is denoted by B_t, the amount of interest earned in the tth coupon by I_t, and the amount of principal adjustment in the tth coupon by P_t. The level coupon will continue to be denoted by Fr. Note that the price P is equal to B_0, while the redemption value C is equal to B_n.

A *bond amortization schedule* is a table which shows the division of each coupon into its interest earned and principal adjustment portions, together with the book value after each coupon is paid. Consider a bond for which C is equal to 1, the coupon is equal to g, and the price is equal to $1 + p$ (where p can be either positive or negative). Table 7.1 is a bond amortization schedule for this bond.

Consider the first coupon period of the bond. At the end of the first coupon period the interest earned on the balance at the beginning of the period is

$$I_1 = i\left[1 + (g - i)a_{\overline{n}|i}\right].$$

The rest of the total coupon of g, i.e.

$$P_1 = g - i\left[1 + (g - i)a_{\overline{n}|i}\right] = (g - i)(1 - ia_{\overline{n}|i}) = (g - i)v^n$$

must be used to adjust the book value. The book value at the end of the period equals the book value at the beginning of the period less the principal adjustment amount, i.e.

$$B_1 = \left[1 + (g - i)a_{\overline{n}|i}\right] - (g - i)v^n = 1 + (g - i)a_{\overline{n-1}|i}.$$

The same reasoning applies for each successive line of the schedule.

Several additional observations are possible. First, it should be noted that the book values agree with the price given by formula (7.2) computed at the original yield rate. Second, the sum of the principal adjustment column is equal to p, the amount of premium or discount, as the case may be. Third, the sum of the interest paid column is equal to the difference between the sum of the coupons and the sum of the principal adjustment column. An algebraic proof of this result is left as an exercise. Fourth, the principal adjustment column is a geometric progression with common ratio $1 + i$. Thus, it is a simple matter to find any one principal adjustment amount knowing any other principal adjustment amount and the yield rate.

When a bond is bought at a premium, the book value will gradually be adjusted downward. This process is called *amortization of premium* or *writing down*. In these cases the principal adjustment amount is often called the "amount for amortization of premium."

When a bond is bought at a discount, the book value will gradually be adjusted upward. This process is called *accumulation of discount* or *writing up*. In these cases the principal adjustment amount is often called the "amount for accumulation of discount."

It should be noted that Table 7.1 is based on a value of $C = 1$. The values in the table are proportional for other values of C. In any particular case, the bond amortization schedule can be constructed from basic principles.

Table 7.1 Bond Amortization Schedule for a \$1 n-period Bond with Coupons at Rate g Bought to Yield Rate i.

Period	*Coupon*	*Interest earned*	*Principal adjustment*	*Book value*
0				$1+p=1+(g-i)a_{\overline{n}\rceil i}$
1	g	$i[1+(g-i)a_{\overline{n}\rceil i}]$	$g-i[1+(g-i)a_{\overline{n}\rceil i}]=(g-i)v^{n}$	$[1+(g-i)a_{\overline{n}\rceil i}]-(g-i)v^{n}=1+(g-i)a_{\overline{n-1}\rceil i}$
2	g	$i[1+(g-i)a_{\overline{n-1}\rceil i}]$	$g-i[1+(g-i)a_{\overline{n-1}\rceil i}]=(g-i)v^{n-1}$	$[1+(g-i)a_{\overline{n-1}\rceil i}]-(g-i)v^{n-1}=1+(g-i)a_{\overline{n-2}\rceil i}$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
t	g	$i[1+(g-i)a_{\overline{n-t+1}\rceil i}]$	$g-i[1+(g-i)a_{\overline{n-t+1}\rceil i}]=(g-i)v^{n-t+1}$	$[1+(g-i)a_{\overline{n-t+1}\rceil i}]-(g-i)v^{n-t+1}=1+(g-i)a_{\overline{n-t}\rceil i}$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$n-1$	g	$i[1+(g-i)a_{\overline{2}\rceil i}]$	$g-i[1+(g-i)a_{\overline{2}\rceil i}]=(g-i)v^{2}$	$[1+(g-i)a_{\overline{2}\rceil i}]-(g-i)v^{2}=1+(g-i)a_{\overline{1}\rceil i}$
n	g	$i[1+(g-i)a_{\overline{1}\rceil i}]$	$g-i[1+(g-i)a_{\overline{1}\rceil i}]=(g-i)v$	$[1+(g-i)a_{\overline{1}\rceil i}]-(g-i)v=1$
Total	ng	$ng-p$	$(g-i)a_{\overline{n}\rceil}=p$	

As an example of a bond bought at a premium, consider a \$1000 par value two-year 8% bond with semiannual coupons bought to yield 6% convertible semiannually. The price of the bond is computed to be \$1037.17. The semiannual coupon is \$40. Table 7.2 is a bond amortization schedule for this example.

Table 7.2 Bond Amortization Schedule for a \$1000 Two-Year Bond with 8% Coupons Paid Semiannually Bought to Yield 6% Convertible Semiannually

Half-year	*Coupon*	*Interest earned*	*Amount for amortization of premium*	*Book value*
0				1037.17
1	40.00	31.12	8.88	1028.29
2	40.00	30.85	9.15	1019.14
3	40.00	30.57	9.43	1009.71
4	40.00	30.29	9.71	1000.00
Total	160.00	122.83	37.17	

In the first line of Table 7.2 we have the following calculations. The interest earned portion of the first coupon is

$$I_1 = iB_0 = .03(1037.17) = \$31.12.$$

The principal adjustment portion of the first coupon is

$$P_1 = Fr - I_1 = 40.00 - 31.12 = \$8.88.$$

The book value at the end of the first period is

$$B_1 = B_0 - P_1 = 1037.17 - 8.88 = \$1028.29.$$

Each successive line of the schedule is calculated in similar fashion.

As an example of a bond bought at a discount, consider the same \$1000 par value two-year 8% bond with semiannual coupons bought to yield 10% convertible semiannually. The price of the bond is computed to be \$964.54. The semiannual coupon is \$40. Table 7.3 is a bond amortization schedule for this example. The reader should verify the entries in Table 7.3.

It should be noted that the amounts for accumulation of discount are really the negatives of the numbers shown in Table 7.3, i.e. they are increments to the book value rather than decrements. However, they are usually written as positive numbers to avoid minus signs. Thus, the reader should be careful to ascertain in any bond authorization schedule whether the bond is selling at a

premium or at a discount, so that the entries in the principal adjustment column can be appropriately interpreted.

Table 7.3 Bond Amortization Schedule for a $1000 Two-Year Bond with 8% Coupons Paid Semiannually Bought to Yield 10% Convertible Semiannually

Half-year	*Coupon*	*Interest earned*	*Amount for accumulation of discount*	*Book value*
0				964.54
1	40.00	48.23	8.23	972.77
2	40.00	48.64	8.64	981.41
3	40.00	49.07	9.07	990.48
4	40.00	49.52	9.52	1000.00
Total	160.00	195.46	35.46	

It should also be noted that if it is desired to find the interest earned or principal adjustment portion of any one coupon, it is not necessary to construct the entire bond amortization schedule. The book value at the beginning of the period in question is equal to the price at that point computed at the original yield rate (which will almost certainly differ from the current market price) and can be determined by the methods of Section 7.3. Then that one line of the schedule can be calculated.

The bond amortization schedule discussed in this section is closely related to the amortization schedule described in Chapter 6. Additional insight into the bond amortization schedule can be obtained from the sinking fund method.

For example, in Table 7.2 the investor can be considered to be investing $1037.17 on which there is a semiannual return of $31.12. This leaves $8.88 each period to place into a sinking fund to replace the premium paid for the bond, since there will be a loss of the premium upon redemption. If the sinking fund can be invested at the yield rate, then the balance in the sinking fund at the end of two years is

$$8.88s_{\overline{4}|.03} = (8.88)(4.1836) = \$37.15$$

which is the amount of premium (with .02 roundoff error). Example 7.4 illustrates a situation in which the sinking fund earns a rate of interest different from the yield rate.

Similarly, in Table 7.3 we have

$$8.23s_{\overline{4}|.05} = (8.23)(4.3101) = \$35.47$$

which is the amount of discount (with .01 roundoff error). It should be noted that the sinking fund in this case is a negative sinking fund.

Another method of writing up or writing down the book values of bonds is the *straight line method*. This method does not produce results which are consistent with compound interest. However, the method is very simple to apply.

In the straight line method the book values are linear, grading from $P = B_0$ to $C = B_n$. Thus, the principal adjustment column is constant

$$P_t = \frac{P - C}{n} \text{ for } t = 1, 2, \ldots, n. \tag{7.7}$$

Note that $P_t > 0$ for premium bonds and $P_t < 0$ for discount bonds. The interest earned column also is constant

$$I_t = Fr - P_t \text{ for } t = 1, 2, \ldots, n. \tag{7.8}$$

Some additional properties of the straight line method are explored in the exercises. Clearly, the larger the amount of premium or discount and the longer the term of the bond, the greater the error in using this method.

The straight line method will also be used in Section 7.6 to derive a formula which is useful in connection with the determination of an unknown yield rate on a bond.

Example 7.4 Find the price of a $1000 par value two-year 8% bond with semiannual coupons bought to yield 6% convertible semiannually if the investor can replace the premium by means of a sinking fund earning 5% convertible semiannually.

The bond will sell at a premium, i.e. $P > 1000$. The semiannual coupon is \$40, and the interest earned is $.03P$. The difference is placed into the sinking fund which must accumulate to the amount of premium. Thus, the fundamental equation of value is

$$(40 - .03P)s_{\overline{4}|.025} = P - 1000$$

$$P = \frac{1000 + 40s_{\overline{4}|.025}}{1 + .03s_{\overline{4}|.025}} = \frac{1000 + (40)(4.1525)}{1 + (.03)(4.1525)} = \$1036.93 .$$

This price is smaller than the bond in Table 7.2. The reader should justify the relative magnitude of the two prices from general reasoning. In general, the price of a bond under these conditions is given by

$$P = \frac{C\left(1 + gs_{\overline{n}|j}\right)}{1 + is_{\overline{n}|j}} \tag{7.9}$$

where i is the yield rate of interest and j the sinking fund rate of interest.

7.5 VALUATION BETWEEN COUPON PAYMENT DATES

The preceding sections have assumed that the price or book value of a bond is being calculated just after a coupon has been paid. It remains to consider the determination of prices and book values between coupon payment dates.

Let B_t and B_{t+1} be the prices or book values of a bond on two consecutive coupon payment dates. Let Fr be the amount of the coupon. As an exercise, the reader will be asked to prove the recursion formula

$$B_{t+1} = B_t(1 + i) - Fr \tag{7.10}$$

assuming a constant yield rate i over this interval. We need to analyze the behavior of B_{t+k} for $0 < k < 1$.

When a bond is bought between coupon payment dates, it is necessary to allocate the coupon for the current period between the prior owner and the new owner. Since the new owner will receive the entire coupon at the end of the period, the purchase price should include a payment to the prior owner for the portion of the coupon attributable to the period from the prior coupon payment date to the date of purchase. This value is called the *accrued coupon* and is denoted by Fr_k. Clearly, at the end points of the interval we would have $Fr_0 = 0$ and $Fr_1 = Fr$, where Fr_1 is computed just before the coupon is paid.

We define the *flat price* of a bond as the money which actually changes hands at the date of sale (ignoring expenses) and denote it by B^f_{t+k}. We define the *market price* of a bond as the price excluding accrued coupon and denote it by B^m_{t+k}. It is clear that

$$B^f_{t+k} = B^m_{t+k} + Fr_k \text{ for } 0 < k < 1. \tag{7.11}$$

In practice, bond prices are quoted as market price plus accrued coupon. As we shall see, this leads to a smooth progression of market prices (assuming a constant yield rate i).

The relationship between the flat price and the market price is illustrated in Figure 7.1. This figure uses the book values for the bond in Table 7.3 as prices. The flat price is denoted by the solid line, while the market price is denoted by the dotted line. The accrued coupon at any date is equal to the vertical distance between the solid line and the dotted line.

The book value of a bond is the asset value assigned to a bond after its purchase. It is common practice for book values to be equal to market values computed at the original yield rate at date of purchase. Of course, these values are quite likely to differ from the current market price if the bond were to be purchased anew. If this practice is followed, then any accrued coupon must be handled as a separate item in the financial statements of the owner of the bond.

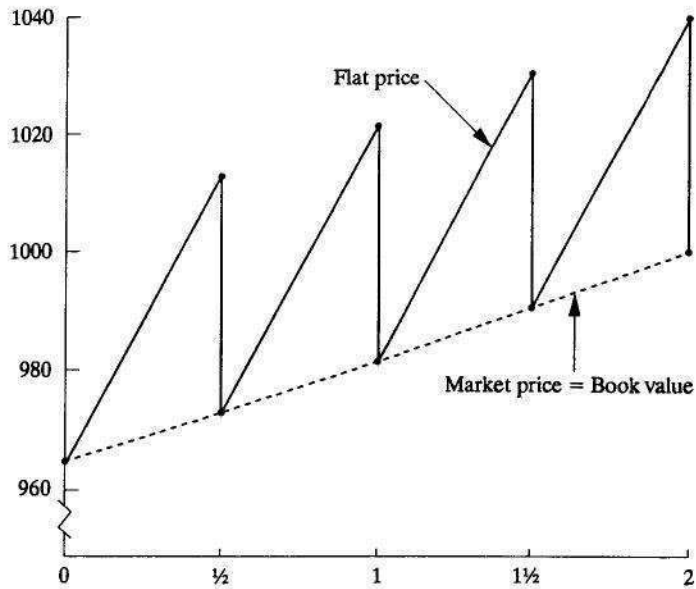

Figure 7.1 Comparison of flat price and market price

There are three methods used to compute the values in formula (7.11). Note that the values on the coupon payment dates are known, so that differences among the three methods arise only for the interim values between coupon payment dates.

The first method is an exact method based on compound interest and is often called the *theoretical method.* The flat price on the interim date is the value on the preceding coupon date accumulated with compound interest at the yield rate for the fractional period

$$B_{t+k}^{f} = B_t(1+i)^k. \tag{7.12}$$

The accrued coupon is computed as in formula (3.22)

$$Fr_k = Fr\left[\frac{(1+i)^k - 1}{i}\right]. \tag{7.13}$$

The market price or book value then is the difference

$$B_{t+k}^{m} = B_t(1+i)^k - Fr\left[\frac{(1+i)^k - 1}{i}\right]. \tag{7.14}$$

One final issue to consider is the amount of premium or discount between coupon payment dates. It is evident that these values should be based on the market price or book value rather than on the flat price. For example, a \$1000 par value bond with a market value of \$980 and accrued coupon of \$30 is still a discount bond despite the fact it is selling for \$1010.

Thus we have

$$\text{Premium} = B^m_{t+k} - C \text{ if } g > i \tag{7.21}$$

and

$$\text{Discount} = C - B^m_{t+k} \text{ if } i > g \tag{7.22}$$

for which B^m_{t+k} can be calculated by any of the three methods described above.

Example 7.5 Compute the flat price, accrued interest, and market price (book value) five months after purchase for the bond in Table 7.2. Use all three methods.

For convenience in illustrating the three methods, we will set $k = 5/6$. For the theoretical method, we have:

$$B^f_{5/6} = 1037.17(1.03)^{5/6} = \$1063.04$$

$$Fr_{5/6} = 40\left[\frac{(1.03)^{5/6} - 1}{.03}\right] = \$33.25$$

$$B^m_{5/6} = 1063.04 - 33.25 = \$1029.79.$$

For the practical method, we have:

$$B^f_{5/6} = 1037.17\left[1 + \frac{5}{6}(.03)\right] = \$1063.10$$

$$Fr_{5/6} = \frac{5}{6}(40) = \$33.33$$

$$B^m_{5/6} = 1063.10 - 33.33 = \$1029.77.$$

For the semi-theoretical method, we have:

$$B^f_{5/6} = \$1063.04$$
$$Fr_{5/6} = \$33.33$$
$$B^m_{5/6} = \$1029.71.$$

In practice, an actual day count would be used based on the calendar date of purchase. For sake of illustration, assume that the bond has a January 1 issue and redemption date and is being purchased on June 1. A pocket calculator with built-in financial functions gives the following answers to this question:

$$B^f_k = \$1063.06$$
$$Fr_k = \$33.37$$
$$B^m_k = \$1029.69.$$

The number of days in the first five months of the year is 151, while the number of days in the first six months of the year is 181. The reader should verify that using $k = 151/181$ in the formulas for the semi-theoretical method produces the above answers. Thus, the day count basis is actual/actual.

7.6 DETERMINATION OF YIELD RATES

As mentioned in Section 7.1, one of the three primary questions under consideration in this chapter is the determination of the yield rate to an investor, given the purchase price of a security. This section discusses this question for bonds.

The determination of the yield to maturity on a bond is similar to the determination of an unknown rate of interest for an annuity, discussed in Section 3.8. We first consider the determination of the yield rate for a bond purchased on a coupon payment date immediately after the coupon is paid.

One approach is linear interpolation in *bond tables*. These are tables of bond prices for a wide variety of terms, coupon rates, and yield rates. Extensive tables of this type have been published. This method is not as widely used as it once was due to the widespread availability of computers with financial software and pocket calculators with built-in financial functions. Nevertheless, bond tables are still used in practice. We will not use this method in this book.

Another approach is to develop approximate formulas for the yield rate by an algebraic technique. We start with formula (7.2)

$$\begin{aligned} P &= C + (Fr - Ci)a_{\overline{n}|} \\ &= C + C(g - i)a_{\overline{n}|} \end{aligned} \tag{7.2}$$

and letting $k = \dfrac{P - C}{C}$ we have

$$(g - i)a_{\overline{n}|} = \frac{P - C}{C} = k$$

or

$$i = g - \frac{k}{a_{\overline{n}|}}. \tag{7.23}$$

To solve formula (7.23) for i, we can use formula (3.25)

$$\frac{1}{a_{\overline{n}|}} = \frac{1}{n}\left[1 + \frac{n+1}{2}i + \frac{n^2 - 1}{12}i^2 + \cdots\right]. \tag{3.25}$$

functions are able to handle such iterations directly. This approach will be illustrated in Example 7.7.

Thus far in Chapter 7 we have computed the yield rate i ignoring reinvestment rates. However, investors in bonds need to consider the rates at which coupons from the bonds can be reinvested. If the investor can only reinvest the coupons at a lower rate than i, then i overstates the yield rate actually realized on the transaction taking into account reinvestment rates. However if the investor is fortunate enough to be able to reinvest the coupons at a higher rate than i, then the overall yield rate on the transaction will actually exceed i.

Consider the situation in which a bond is purchased for P, coupons of Fr are paid at the end of each period for n periods, the bond is redeemed for C at the end of n periods, and the coupons are reinvested at rate j. If we denote the yield rate considering reinvestment by i' (to distinguish it from i), then i' would satisfy the following equation of value

$$P(1 + i')^n = Frs_{\overline{n}|j} + C \tag{7.27}$$

since both sides represent the value of the investment at the end of n periods. An illustration of yield rates taking into account reinvestment rates is given in Example 7.8.

One of the reasons for the popularity of zero coupon bonds in recent years is that such bonds do not present a reinvestment risk to the investor. Since there are no coupons to reinvest, the yield rate is locked-in at the date of purchase.

One final consideration in practical applications is the effect of expenses. If commissions and other fees are involved in buying and selling bonds, then the actual yield rate realized will be reduced. However, the procedures described above can readily be applied to reflect such expenses. This would be achieved by increasing the purchase price by any expenses at purchase and reducing the sale price by any expenses at sale. It should be noted that generally no expenses are charged when a bond is redeemed or matures.

Example 7.6 A \$100 par value 10-year bond with 8% semiannual coupons is selling for \$90. Find the yield rate convertible semiannually.

We will illustrate the methods discussed in this section. We have

$$k = \frac{P - C}{C} = \frac{90 - 100}{100} = -.1.$$

The bond salesman's method formula (7.25), gives the following approximate semiannual yield rate

$$i = \frac{.04 + \frac{.1}{20}}{1 + (.5)(-.1)} = .0474$$

i.e. 4.74%, or 9.48% convertible semiannually.

The more refined formula (7.24) gives

$$i = \frac{.04 + \frac{.1}{20}}{1 + \left[\frac{21}{40}\right](-.1)} = .0475$$

i.e. 4.75%, or 9.50% convertible semiannually, which should be closer to the true yield rate.

We next illustrate an ad hoc iteration with formula (7.23) using a starting value $i_0 = .0475$. We obtain the following series of values:

$$\begin{aligned} i_0 &= .0475 \\ i_1 &= .04786 \\ i_2 &= .04788 \\ i_3 &= .04788 \end{aligned}$$

Thus, the yield rate convertible semiannually is 2(.04788) = .09576, or 9.576%. Formula (7.24) is indeed more accurate than formula (7.25), as expected. The rate of convergence using this ad hoc method is quite acceptable.

Finally, we use the Newton-Raphson method with formula (7.26). Since this method is so powerful, we will carry the calculation to two more decimal places of accuracy. We obtain the following series of values:

$$\begin{aligned} i_0 &= .0475 \\ i_1 &= .04788 \\ i_2 &= .0478807 \\ i_3 &= .0478807 \end{aligned}$$

Thus, the yield rate convertible semiannually is 2(.0478807) = .0957614, or 9.57614%. This answer was confirmed on a pocket calculator with built-in financial functions.

Example 7.7 Assume that the bond in Example 7.6 was issued on March 1. Slightly over two years later on May 15 the market price of the bond is 88, compute the yield rate if the bond is bought on that date.

A pocket calculator with built-in financial functions produces the answer 10.2694%. We will attempt to confirm this answer. The price of the bond on the immediately preceding March 1 at a yield rate of 10.2694% convertible semiannually is computed to be 87.8194. The number of days from March 1 to May 15 is 75, while the number of days from March 1 to September 1 is 184. Using formula (7.20), the semi-theoretical method produces

$$87.8194(1.051347)^{75/184} - \frac{75}{184} \cdot 4 = 87.9998$$

It is evident that the lowest price in each category will result when the lowest valu of n is used, i.e. $n = 10$, 20, and 30. Thus, the prices to compare are:

$$109.00 + (.3650)a_{\overline{10}|.015} = 109.00 + (.3650)(9.2222) = \$112.37$$
$$104.50 + (.4325)a_{\overline{20}|.015} = 104.50 + (.4325)(17.1686) = \$111.93$$
$$100.00 + (.5000)a_{\overline{30}|.015} = 100.00 + (.5000)(24.0158) = \$112.01.$$

The lowest price \$111.93 occurs for $n = 20$, i.e. for redemption 10 years af issue. This example illustrates that the least favorable redemption date can occ between the earliest and latest possible redemption dates when the redemption valu are not equal at all the possible redemption dates. It is fairly common in practic for a callable bond to have redemption values which decrease as the term of the bon increases, as in this example. The excess of the redemption value over the pa value, i.e. \$9 and \$4.50 in this example, is often termed the *call premium*.

7.8 SERIAL BONDS

As mentioned in Section 7.2, a borrower in need of a large amount of fund may issue a series of bonds with staggered redemption dates instead of using common maturity date. These types of bonds are called *serial bonds* and ar considered further in this section.

If the redemption date of each individual bond is known, then the valuatio of any one bond can be performed by methods already described. The value o the entire issue of bonds is merely the sum of the values of the individual bonds It will be shown that the value of the entire issue of bonds can often be foun more efficiently than by summing the individual bond values, however.

In some cases, the bonds to be redeemed on each successive redemptio date are not known in advance but are chosen randomly. The value of any on bond is impossible to determine in advance with certainty, since its redemptio date depends upon chance. However, the value of the entire issue of bonds ca be determined with certainty.

Of course, values for any one bond can be determined on the same basis a that used for callable bonds in Section 7.7. Also, the yield rate actually realize after redemption can be determined.

The valuation of an issue of serial bonds can most efficiently be performe using Makeham's formula

$$P = K + \frac{g}{i}(C - K). \tag{7.4}$$

Assume that the serial bonds are redeemable at m different redemption dates Let the purchase price, redemption value, and present value of the redemptio value for the first redemption date be denoted by P_1, C_1, and K_1; for th second redemption date by P_2, C_2, and K_2; and so forth until the las redemption date when we have P_m, C_m, and K_m. Thus we have

$$P_1 = K_1 + \frac{g}{i}(C_1 - K_1)$$
$$P_2 = K_2 + \frac{g}{i}(C_2 - K_2)$$
$$\vdots \qquad \vdots$$
$$P_m = K_m + \frac{g}{i}(C_m - K_m)$$

summing, we obtain

$$P' = K' + \frac{g}{i}(C' - K') \tag{7.28}$$

ere

$$P' = \sum_{t=1}^{m} P_t$$
$$C' = \sum_{t=1}^{m} C_t$$
$$K' = \sum_{t=1}^{m} K_t.$$

e reader should note that the symbol P_t was also used in Section 7.4 to resent the amount of principal adjustment in the tth bond coupon.

Thus, the price of an entire issue of serial bonds is denoted by P' and is en by formula (7.28). This formula is usually more efficient than a nmation of the prices of each individual bond, since C' and K' are usually te simple to obtain. The sum C' is merely the sum of the redemption ues for the entire issue of bonds, a figure readily obtainable. The sum K' is simple form if the bonds are redeemed according to a systematic pattern at ular intervals. In this case, K' is some form of annuity for which a simple pression is possible. Example 7.10 illustrates this technique.

Example 7.10 Find the price of a $1000 issue of 5 1/4% bonds with nual coupons which will be redeemed in 10 annual installments at the end the 11th through the 20th years from the issue date at 105. The bonds are ght to yield 7% effective.

In this example we have:

$$F = 100$$
$$C = 105$$
$$r = .0525$$
$$g = \frac{100}{105}(.0525) = .05$$
$$i = .07$$

The present value of the redemption value is

$$1050(1.05)^{-10}(1.045)^{-10} = 415.082.$$

Thus, the price of the bond is 528.334 + 415.082 = \$943.42. The price is higher than in Example 7.3, as would be expected, since the yield rate is lower for the final five years in the term of the bond.

7.10 OTHER SECURITIES

The preceding sections have been limited to a discussion of redeemable bonds. This section is concerned with two other types of securities: (1) preferred stock and perpetual bonds, and (2) common stock.

Preferred stock and perpetual bonds

Preferred stock and perpetual bonds are similar in that they both are types of fixed-income securities without redemption dates. Thus, the price must be equal to the present value of future dividends or coupons forever, i.e. the dividends or coupons form a perpetuity. The version of formula (7.1) for the situation would be

$$P = \frac{Fr}{i}. \tag{7.31}$$

The reader should be aware that some preferred stock is issued with a redemption date. Such preferred stock can be handled exactly like a bond, as described in the preceding sections.

Common stock

Common stock presents a different problem, since it is not a fixed-income security, i.e. the dividends are not known in advance, nor are they level. In practice, common stock prices fluctuate widely in the stock market, often for little apparent reason.

In theory, common stock prices should represent the present value of future dividends. Values computed in this fashion can be characterized as being based on the *dividend discount model.* Of course, this calculation should take into account projected changes in the dividend scale.

Consider a situation in which a corporation is planning to pay a dividend of D at the end of the current period. Assume that dividends are projected to change geometrically with common ratio $1 + k$ indefinitely and that the stock is purchased to yield i per period, where $-1 < k < i$.

Then the theoretical price of the stock is obtained by taking the limit of formula (4.39) as $n \to \infty$, which gives

$$P = D\,\frac{1}{i-k}. \tag{7.32}$$

A theoretical calculation of this type is illustrated in Example 7.13. The reader should note that the symbol D was also used in Section 6.4 to represent a sinking fund deposit.

It is probably unrealistic to project constant percentage increases in dividends indefinitely into the future. As corporations increase in size and become more mature, the rate of growth will generally slow down. An illustration of the dividend discount model applied under these conditions is given in Example 7.14.

In practice, the most common frequency of dividend payments on both preferred and common stock in the United States is quarterly. This is in contrast to the typical semiannual frequency of coupon payments on bonds.

Example 7.13 ***A common stock is currently earning $4 per share and will pay $2 per share in dividends at the end of the current year. Assuming that the earnings of the corporation increase 5% per year indefinitely and that the corporation plans to continue to pay 50% of its earnings as dividends, find the theoretical price to earn an investor an annual effective yield rate of: (1) 10%, (2) 8%, and (3) 6%.***

1. The present value of dividends is

$$2\left[\frac{1}{.10-.05}\right] = \$40.$$

Thus, the theoretical price is 10 times current earnings.

2. The present value of dividends is

$$2\left[\frac{1}{.08-.05}\right] = \$66\,2/3.$$

Thus, the theoretical price is 16 2/3 times current earnings.

3. The present value of dividends is

$$2\left[\frac{1}{.06-.05}\right] = \$200.$$

Thus, the theoretical price is 50 times current earnings.

Example 7.14 ***Rework Example 7.13 assuming that the rate of increase in earnings is 5% for the first five years, 2 1/2% for the second five years, and 0% thereafter. Assume an annual effective yield rate of 10%.***

Successive applications of formula (4.39) give

$$2\left[\frac{1-\left[\frac{1.05}{1.10}\right]^5}{.10-.05}\right] + 2\frac{(1.05)^5}{(1.10)^5}\left[\frac{1-\left[\frac{1.025}{1.10}\right]^5}{.10-.025}\right] + \frac{2(1.05)^5(1.025)^5}{(1.10)^{10}}\cdot\frac{1}{.10} = \$25.72$$

in calculating many of their liability values. The relationship between assets and liabilities is an important issue which is discussed further in Chapter 9.

The present value method is very sensitive to the choice of interest rate used in taking present values. This can be both an advantage and a disadvantage of the method. The method is flexible, but does have a degree of arbitrariness in connection with the choice of the interest rate to use in computing present values. The present value method can produce asset values which are significantly different from either market value or book value. Also the method is less easily understood than either market value or book value.

In summary, there is no valuation method for securities which is used in all situations. The reader should be careful when encountering asset values in practical situations to ascertain the method of valuation.

The choice of valuation methods for securities will affect computed yield rates. For example, in Section 5.5 it was noted that formula (5.14)

$$i \fallingdotseq \frac{2I}{A + B - I} \tag{5.14}$$

is often used to compute the yield rate earned by an investment fund. The values of A and B are dependent upon the asset valuation methods used. Also, the value of I may vary substantially depending upon whether or not capital gains are included in I.

EXERCISES

7.2 Types of securities

1. Find the price which should be paid for a zero coupon bond which matures for \$1000 in 10 years to yield:
   a) 10% effective.
   b) 9% effective.
   c) Thus, a 10% reduction in the yield rate causes the price to increase by what percentage?

2. A 10-year accumulation bond with an initial par value of \$1000 earns interest of 8% compounded semiannually. Find the price to yield an investor 10% effective.

3. A 26-week T-bill is bought for \$9600 at issue and will mature for \$10,000. Find the yield rate computed as:
   a) A discount rate, using the typical method for counting days on a T-bill.
   b) An annual effective rate of interest, assuming the investment period is exactly half a year.

4. T-bills of all maturities yield 8% computed on a discount basis. Find the ratio of the annual effective rate of interest earned on a 52-week T-bill to that earned on a 13-week T-bill. Use an approach which does not involve the counting of days.

7.3 Price of a bond

5. A 10-year \$100 par value bond bearing a 10% coupon rate payable semiannually and redeemable at \$105 is bought to yield 8% convertible semiannually. Find the price. Verify that all four formulas produce the same answer.

6. For the bond in Example 7.3, determine the following:
   a) Nominal yield, based on the par value.
   b) Nominal yield, based on the redemption value.
   c) Current yield.
   d) Yield to maturity.

7. Two \$100 par value bonds both with 8% coupon rates payable semiannually are currently selling at par. Bond A matures in 5 years at par, while Bond B matures in 10 years at par. If prevailing market rates of interest suddenly go to 10% convertible semiannually, find the percentage change in the price of:
   a) Bond A.
   b) Bond B.
   c) Justify from general reasoning the relative magnitude of the answers to (*a*) and (*b*).

8. Two \$1000 bonds redeemable at par at the end of the same period are bought to yield 4% convertible semiannually. One bond costs \$1136.78 and has a coupon rate of 5% payable semiannually. The other bond has a coupon rate of 2 1/2% payable semiannually. Find the price of the second bond.

9. A \$1000 bond with a coupon rate of 9% payable semiannually is redeemable after an unspecified number of years at \$1125. The bond is bought to yield 10% convertible semiannually. If the present value of the redemption value is \$225 at this yield rate, find the purchase price.

10. A \$1000 par value n-year bond maturing at par with \$100 annual coupons is purchased for \$1110. If $K = 450$, find the base amount G.

11. An investor owns a \$1000 par value 10% bond with semiannual coupons. The bond will mature at par at the end of 10 years. The investor decides that an 8-year bond would be preferable. Current yield rates are 7% convertible semiannually. The investor uses the proceeds from the sale of the 10% bond to purchase a 6% bond with semiannual coupons, maturing at par at the end of 8 years. Find the par value of the 8-year bond.

12. An n-year \$1000 par value bond matures at par and has a coupon rate of 12% convertible semiannually. It is bought at a price to yield 10% convertible

7.8 Serial bonds

36. A \$10,000 serial bond is to be redeemed in \$1000 installments of principal per half-year over the next five years. Interest at the annual rate of 12% is paid semiannually on the balance outstanding. How much should an investor pay for this bond in order to produce a yield rate of 8% convertible semiannually?

37. A \$10,000 serial bond is to be redeemed in \$500 installments of principal at the end of the 6th through the 25th years from the date of issue. Interest at the rate of 6% is paid annually on the balance outstanding. What is the price to yield an investor 10% effective?

38. Find an expression for the present value of a \$100,000 issue of serial bonds, if it is known that the yield rate is 125% of the coupon rate and that the bonds are redeemable at par according to the following schedule:

End of Years	Amount Redeemable
5, 8, 11	\$10,000
14, 17	20,000
20	30,000

All rates are semiannual. Express your answer strictly as a function of $a_{\overline{n}|}$'s for various values of n.

39. A \$78,000 issue of serial bonds with annual coupons of 4% on the balance outstanding is to be redeemed in 12 annual installments beginning at the end of the 5th year. The amount redeemed at the end of the 5th year is \$12,000, at the end of the 6th year \$11,000, and so forth, until all the bonds are redeemed. Find an expression for the price to yield an investor 5% effective.

7.9 Some generalizations

40. Derive the following variations of formula (7.29):

a) $P = C + \left(\frac{Fr}{s_{\overline{k}|}} - Ci \right) a_{\overline{n}|}$.

b) $P = \frac{G}{s_{\overline{k}|}} + \left(C - \frac{G}{s_{\overline{k}|}} \right) v^n$.

c) $P = K + \frac{g}{is_{\overline{k}|}}(C - K)$.

41. Derive the following variations of formula (7.30):

a) $P = C + (Fr - Ci^{(m)})\, a_{\overline{n}|}^{(m)}$.

b) $P = Gs_{\overline{1}|}^{(m)} + (C - Gs_{\overline{1}|}^{(m)})\, v^n$.

c) $P = K + \frac{g}{i^{(m)}}(C - K)$.

42. Find the price of a \$1000 par value 10-year bond maturing at par which has coupons at 8% per annum payable quarterly and is bought to yield 6% per annum convertible semiannually.

43. The price of a \$100 bond, which matures in n years for \$105, has semiannual coupons of \$4, and is bought to yield an effective rate i, can be expressed as

$$\frac{Av^n + B}{i^{(2)}}.$$

Find A and B.

44. A \$1000 par value 20-year bond maturing at par has annual coupons of 5% for the first 10 years and 4% for the second 10 years. Find an expression for the price of the bond bought to yield a rate of interest convertible quarterly.

45. A 10-year bond has annual coupons which vary $10, 9, 8, \ldots, 1$, and it matures for \$100. If the bond is bought to yield i effective, find expressions for:
   a) The amount of the fifth coupon which is required interest.
   b) The amount of the fifth coupon used to amortize the book value.

46. An investor buys a \$100 par value 3% bond with semiannual coupons on which \$50 matures in 9 years at \$51 and the other \$50 matures in 10 years at \$50. Show that the price to yield an effective rate of interest i is

$$3a_{\overline{10}|}^{(2)} + (101 + 51i - 1.5s_{\overline{1}|}^{(2)})v^{10}.$$

7.10 Other securities

47. What would be the answer to Example 7.13 for a yield rate of 5% or less?

48. A preferred stock pays a \$10 dividend at the end of the first year, with each successive annual dividend being 5% greater than the preceding one. What level annual dividend would be equivalent if $i = 12\%$?

49. A common stock pays annual dividends at the end of each year. The earnings per share in the year just ended were \$6. Earnings are assumed to grow 8% per year in the future. The percentage of earnings paid out as a dividend will be 0% for the next 5 years and 50% thereafter. Find the theoretical price of the stock to yield an investor 15% effective.

50. A common stock is purchased at a price equal to 10 times current earnings. During the next 6 years the stock pays no dividends, but earnings increase 60%. At the end of 6 years the stock is sold at a price equal to 15 times earnings. Find the effective annual yield rate earned on this investment.

51. Find an expression for the theoretical price of a common stock paying annual dividends at the end of each year. The earnings in the year just ending were E. It is assumed that the rate of growth in earnings for the tth year is k_t, the yield rate

8

Practical applications

8.1 INTRODUCTION

Chapter 8 contains several practical applications of the theory of interest not previously discussed. Sections 8.2 through 8.4 cover the area of consumer credit, including a discussion of "truth in lending" requirements.

Section 8.5 and 8.6 deal with topics involved in the financial analysis of investments in fixed assets which have not previously been discussed. Section 8.5 introduces the reader to various depreciation methods which are in common usage. Section 8.6 discusses a type of financial analysis known as capitalized cost, a concept related to capital budgeting which was discussed in Section 5.8.

Section 8.7 analyzes the calculation of yield rates on short sales, a type of financial transaction not previously considered. Some unusual difficulties arise in connection with such calculations.

Finally, Section 8.8 provides an overview of the wide variety of modern financial instruments which are in existence today. This material is of considerable value in introducing the reader to a number of investment vehicles which may be unfamiliar. It also provides the foundation for further analytical development in Chapters 9 and 10.

8.2 TRUTH IN LENDING

In 1968 the United States Congress enacted the Consumer Credit Protection Act. Title I of this law is widely known as the "Truth in Lending Act" (15 USC 1601).

The primary purpose of this law is to require lenders to provide fair and accurate disclosure of the terms of consumer loans to borrowers. The law does not attempt to control the amount which lenders may charge on loans, but only

to require proper disclosure. Also, the law applies only to consumer loans, and not business loans.

The law requires the disclosure of two key financial values; namely, the *finance charge* and the *annual percentage rate* (often called the "APR"). The former is designed to express the amount of interest to be charged over the term of the loan in dollars, while the latter expresses the interest to be paid as an annual rate. In addition to these two financial values, a number of additional disclosures in narrative form are required.

At the time a loan is taken out, the borrower may incur certain charges in connection with obtaining the loan. Examples of such charges would be points (to be defined in Section 8.3), other loan fees, service charges, credit report fees, and premiums for credit insurance.

The finance charge, as defined in the Truth in Lending Act, includes all interest charges. Moreover, it also includes some, but not necessarily all, of the additional charges cited in the preceding paragraph. Detailed regulations have been released by the Federal Reserve Board, the federal agency responsible for administering the Truth in Lending Act, which define items that must be included in the finance charge and those that do not have to be included.

It is specified that the annual percentage rate is to be computed by the *actuarial method*. This method is defined on a basis compatible with compound interest theory. Using the actuarial method leads to a subdivision of payments between principal and interest which is consistent with the amortization schedule discussed in Section 6.3.

One interesting aspect of the annual percentage rate is that it is quoted as a nominal rate convertible at the frequency with which payments are being made, rather than as an effective rate. For example, two loans might both quote "APR = 12%," but if one is repaid with monthly installments and the other is repaid with quarterly installments, the two rates are not equivalent. Thus, annual percentage rates on various loans cannot validly be compared directly unless they are all at the same frequency. The length of the payment interval is called the *unit period* in the law.

Truth in lending requirements make a distinction between open end credit and closed end credit. Examples of *open end credit* are a revolving charge account or a credit card. For open end credit the finance charge need only be disclosed each period as it is computed. The annual percentage rate is merely the nominal annual rate being charged on the outstanding loan balances. For example, if a credit card company charges interest each month at 1.75% on the outstanding balance, it must quote "APR = 21%."

The above brief discussion is intended to give the reader a general appreciation of the Truth in Lending Act and its application under typical conditions. However, the discussion is far from complete. Readers interested in learning more details about truth in lending are referred to "Regulation Z Truth in Lending" published by the Federal Reserve Board (12 CFR 226). An "Official Staff Commentary on Regulation Z Truth in Lending" has also been published to elaborate upon the formal regulation.

Some historical perspective on the computation of interest rates on loans is instructive. Until the early 1800s the most common method was the *Merchant's Rule*, which is essentially equivalent to simple interest. The Merchant's Rule worked satisfactorily for short-term loans, but produced illogical results for longer-term loans.

The issue reached the courts in Virginia in 1795. The landmark Ross v. Pleasants case involved a dramatic result. The roles of the debtor and creditor were reversed when the Merchant's Rule was applied over an extended period of time. Needless to say, the original creditor who became a debtor was none too pleased at this turn of events!

The issue was finally resolved by the United States Supreme Court in the decision on Story v. Livingston (38 U.S. 359) 1839. The court decided that payments by the borrower should first be applied to pay any accrued interest, with any excess being applied to reduce the outstanding loan balance. This decision is consistent with the amortization scheduled developed in Section 6.3 and with compound interest theory. In recognition of this Supreme Court decision, the above procedure for dividing installment payments between interest and principal is often called the *United States Rule*, a term used in contrast to the Merchant's Rule.

The following is an extract from the decision in the Story v. Livingston case written by Mr. Justice Wayne:

> "The correct rule in general is that the creditor shall calculate interest whenever a payment is made. To this interest the payment is first to be applied; and if it exceed the interest due, the balance is to be applied to diminish the principal. If the payment fall short of the interest, the balance of interest is not to be added to the principal, so as to produce interest. Thus rule is equally applicable whether the debt be one which expressly draws interest, or one in which interest is given in the name of damages."

Two aspects of this decision are worthy of comment. First, the United States Rule will result in a compounding of interest whenever an installment payment is made. If installment payments are made at a regular frequency, then

interest will be convertible at the same regular frequency as payments are made. However, if installment payments are made at an irregular frequency, then the United States Rule produces the rather odd result that the rate of interest quoted is convertible at the same irregular frequency as payments are made. This result is illustrated in Example 8.3.

Second, it is often said that the United States Rule and the actuarial method are equivalent. Under most circumstances, they are equivalent. However, the United States Rule is not consistent with the actuarial method when a payment is made that is insufficient to cover the accrued interest. Under the United States Rule any such deficiency is not added to the outstanding loan balance to accrue additional interest. However, under the actuarial method, which is consistent with compound interest theory, any such deficiency must be capitalized, i.e. added to the outstanding loan balance to accrue additional interest.

Example 8.1 Find the annual percentage rate (APR) on a consumer loan of $1000 which is repaid with monthly installments of $90 at the end of each month for one year.

The finance charge is equal to

$$K = (12)(90) - 1000 = 80.$$

The monthly rate j is determined from an application of formula (8.2)

$$90a_{\overline{12}|j} = 1000$$

or

$$a_{\overline{12}|j} = 11.1111.$$

The Federal Reserve Board tables promulgated under Regulation Z give the answer $12j = 14.50\%$ for this example. The tables contain answers accurate only to the nearest .25%, which is the minimum level of accuracy required under truth in lending.

A more accurate answer can be obtained using iteration for an unknown rate of interest as discussed in Section 3.8. Such an iteration produces $j = .012043$, so that the APR $= 12j = .1445$, or 14.45%.

Prior to the passage of the Truth in Lending Act, this type of loan arrangement was frequently called an "8% add-on," since the finance charge for one year was 8% of the loan amount. This misleading statement implied a much lower rate of interest than the true rate given by the APR. It was this type of widespread misrepresentation that led to the enactment of truth in lending.

Example 8.2 A purchaser of a new automobile needs to finance $10,000 of the purchase price. The dealer offers two options for financing the loan with monthly payments over four years. Under Option A the APR is 9%. Under Option B the dealer offers a "cash back" of $600 and an APR of 12%.

We define L, K, R, n, i, and j as in Section 8.2 and set $m = 12$. We then make the following additional definitions:

Q = expenses at settlement that must be reflected in the APR
L^* = amount financed for truth in lending purposes, i.e. reflecting Q
j' = monthly rate of interest on the loan
i' = quoted annual rate of interest on the loan

We then have the following relationships. The quoted annual rate of interest on the loan is

$$i' = 12j'. \tag{8.4}$$

The monthly payment on the loan is

$$R = \frac{L}{a_{\overline{n}|j'}}. \tag{8.5}$$

The amount financed for truth in lending purposes is the amount of the loan less the expenses at settlement that must be reflected, i.e.

$$L^* = L - Q. \tag{8.6}$$

The finance charge is equal to the difference between the total payments to be made and the amount financed for truth in lending purposes, i.e.

$$K = nR - L^*. \tag{8.7}$$

To obtain the rate of interest per month for truth in lending we solve the following equation for j

$$Ra_{\overline{n}|j} = L^*. \tag{8.8}$$

Finally, the annual percentage rate for truth in lending is

$$i = 12j. \tag{8.9}$$

The loan amortization schedules considered in Chapter 6 involved loans with relatively short terms. The reader will find it instructive to examine an illustrative 30-year amortization schedule for a real estate mortgage contained in Appendix VII. The early payments are almost entirely interest, while the ending payments are almost entirely principal. Many people find it frustrating to buy a home, make payments for several years, and then find that the outstanding loan balance on their real estate mortgage has declined very little.

In recent years a new type of mortgage loan has appeared, the *adjustable rate mortgage*. This term is in contrast to the traditional mortgage, which is often called a *fixed rate mortgage*. Under the adjustable rate mortgage the rate of interest charged on the loan can be periodically adjusted upward or downward by the lender under certain conditions and subject to certain restrictions.

The motivating factor for the development of the adjustable rate mortgage is the fact that lenders are willing to provide such mortgages at lower initial rates of interest than on traditional fixed rate mortgages. This reaction by lenders is understandable, since lenders are making firm commitments on the rate of interest to be charged on fixed rate mortgages over periods ranging from 15 to 30 years. In periods of rising interest rates lenders will be locked into fixed rate portfolios with yield rates well below prevailing market rates. On the other hand, in periods of falling interest rates borrowers will be able to refinance their loans at the lower rates.

From the borrower's perspective the adjustable rate mortgage may be attractive, since the initial rate of interest and the initial monthly payment are lower than with a fixed rate mortgage. Moreover, if interest rates fall, the monthly payment will be adjusted even lower.

However, there is a significant risk with an adjustable rate mortgage that rates of interest and monthly payments will increase and exceed those that would have existed had a fixed rate mortgage been chosen originally. Thus, with a fixed rate mortgage the risk associated with interest rate fluctuations is borne by the lender, whereas with adjustable rate mortgages much of that risk is shifted to the borrower.

The interval at which the lender can adjust the interest rate on an adjustable rate mortgage is called the *adjustment period*. This interval is typically every one, three, or five years.

The interest rates on an adjustable rate mortgage are usually tied by formula to an *index rate*. This prevents the potential abuse of the lender making arbitrary upward adjustments in the interest rate even if prevailing market rates would not justify such increases. One example of a possible index rate is the yield on Treasury securities of a specified maturity. A second example would be an index rate based on the interest rates being paid to depositors in order to secure funds to lend by a broad cross section of institutional lenders. The interest rate on an adjustable rate mortgage is computed as the index rate plus a fixed *margin*, where the margin is defined by the original mortgage agreement.

Most adjustable rate mortgages have limitations on the amount of increase in the interest rate and/or payments which are called *caps*. One type of cap is the *interest rate cap*. This type of cap places a limit on the amount of any periodic increase in the interest rate, or a limit on the total increase over the life of the loan, or both. A second type of cap is the *payment cap*. This type of cap places a percentage limit on the dollar increase in monthly payment that can occur at any one adjustment date.

Note that the total payments over 30 years are more than three times the amount of the loan! The effect of compound interest over long periods of time is truly dramatic.

Finally, we need to compute the APR. From formula (8.8) we have

$$1044.23a_{\overline{360}|j} = 117{,}800.$$

Using iteration as described in Section 3.8, we find that $j = .008433$. Thus, the APR is

$$\text{APR} = i = 12(.008433) = .1012, \text{ or } 10.12\%.$$

The APR is greater than 9.9%, as expected.

***Example 8.5** A borrower takes out a 30-year adjustable rate mortgage for $65,000. The interest rate for the first year is 8%. If the interest rate increases to 10% for the second year, find the increase in the monthly payment.*

The monthly payment for the first year is

$$\frac{65{,}000}{a_{\overline{360}|.08/12}} = \frac{65{,}000}{136.2835} = \$476.95.$$

The outstanding loan balance after one year is

$$476.95a_{\overline{348}|.08/12} = 476.95(135.1450) = \$64{,}457.42.$$

The revised monthly payment for the second year is

$$\frac{64{,}457.42}{a_{\overline{348}|.10/12}} = \frac{64{,}457.42}{113.3174} = \$568.82.$$

Thus, the increase in the monthly payment is

$$568.82 - 476.95 = \$91.87$$

which is an increase of 19.3%.

8.4 APPROXIMATE METHODS

The calculation of an exact annual percentage rate is somewhat complicated since it involves an iteration. Several approximate methods for calculating the unknown rate of interest on installment loans have been developed. These methods do not utilize any interest functions and can be calculated directly.

Although these methods are not as important as they were in the days before the development of pocket calculators with built-in financial functions, they still have considerable educational and conceptual value. Moreover, they are useful in providing the background necessary for computing the unearned finance charge defined in Section 8.2 and for the various depreciation methods to be considered in Section 8.5. Finally, one of these approximate methods has such a high level of accuracy that it has value in other situations.

We will examine four such approximate methods. All of these methods are similar in that they replace the true division of the installment payments between

principal and interest with an arbitrary division between principal and interest. It is assumed that the amount of principal invested during each mth of a year is invested at rate i/m for that mth of a year. Thus, if we denote the outstanding loan balance at time t/m by $B_{t/m}$, we have

$$\frac{i}{m} \sum_{t=0}^{n-1} B_{t/m} = K$$

or

$$i = \frac{mK}{\sum_{t=0}^{n-1} B_{t/m}}. \tag{8.10}$$

The derivation of formula (8.10) reflects the fact that the finance charge K must equal the sum of the amounts of interest earned during each mth of a year on the outstanding loan balances for those periods. The various methods differ only the arbitrary division of installment payments between principal and interest, which will be reflected in the denominator of formula (8.10).

The first method is often called the *maximum yield method*, since it produces larger answers than any of the other methods. The value of i produced by this method is labelled i^{max}. This method assumes that all installment payments are applied entirely to principal until it is fully paid and thereafter are applied entirely to interest. We also assume that the finance charge is less than one installment payment, i.e.

$$K < \frac{L + K}{n}.$$

This latter assumption leads to the result that all installment payments except part of the last one are used for principal repayment.

With these assumptions Table 8.1 is a modified amortization schedule for this method. The reader should verify the entries in Table 8,1 and note the various relationships in the table.

We now need to sum the outstanding loan balance column to evaluate formula (8.10)

$$\sum_{t=0}^{n-1} B_{t/m} = Ln - \frac{L + K}{n} \cdot \frac{(n-1)n}{2}$$

so that

$$i^{max} = \frac{mK}{Ln - (L + K)\dfrac{n-1}{2}}$$

$$= \frac{2mK}{L(n+1) - K(n-1)}. \tag{8.11}$$

so that

$$i^{cr} = \frac{mK}{L \cdot \dfrac{n+1}{2}} = \frac{2mK}{L(n+1)}. \tag{8.13}$$

Table 8.3 Amortization Schedule Using the Constant Ratio Method

Period	*Payment amount*	*Interest paid*	*Principal repaid*	*Outstanding loan balance*
0				$\frac{n}{n}L = L$
$\frac{1}{m}$	$\frac{L+K}{n}$	$\frac{K}{n}$	$\frac{L}{n}$	$\frac{n-1}{n}L$
$\frac{2}{m}$	$\frac{L+K}{n}$	$\frac{K}{n}$	$\frac{L}{n}$	$\frac{n-2}{n}L$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$\frac{n-1}{m}$	$\frac{L+K}{n}$	$\frac{K}{n}$	$\frac{L}{n}$	$\frac{1}{n}L$
$\frac{n}{m}$	$\frac{L+K}{n}$	$\frac{K}{n}$	$\frac{L}{n}$	0
Total	$L+K$	K	L	

It is interesting to note that formula (8.13) can be derived by an alternative argument. The amount of interest per year is mK/n. The average outstanding loan balance can be found by averaging the outstanding loan balances during the first period and the last period, i.e.

$$\frac{1}{2} \cdot L \cdot \frac{n}{n} + \frac{1}{2} \cdot L \cdot \frac{1}{n} = L \cdot \frac{n+1}{2n}.$$

The rate of interest is computed as the annual amount of interest divided by the average outstanding loan balance, i.e.

$$i^{cr} = \frac{m \cdot \dfrac{K}{n}}{L \cdot \dfrac{n+1}{2n}} = \frac{2mK}{L(n+1)}. \tag{8.13}$$

The two methods are equivalent, since the outstanding loan balance is linear under these assumptions.

On occasion an even simpler version of the constant ratio method is encountered in which the average outstanding loan balance is taken to be the average of the beginning balance and the ending balance, i.e.

$$\frac{1}{2} \cdot L + \frac{1}{2} \cdot 0 = \frac{1}{2} \cdot L .$$

With this assumption the rate of interest is

$$i^{cr'} = \frac{2mK}{Ln} . \tag{8.14}$$

Formula (8.14) replaces the term $(n + 1)/2n$ with $1/2$. The reader should note that the relationship between formulas (8.13) and (8.14) is quite analogous to the relationship between formulas (7.24) and (7.25), the two versions of the bond salesman's formula. In general, formula (8.14) produces less accurate results than formula (8.13).

It is interesting to note that i^{cr} is equal to the harmonic mean of i^{max} and i^{min}. The proof of this result is left as an exercise.

The fourth method is the *direct ratio method*. The value of i produced by this method is labelled i^{dr}. This method uses an approximate division into principal and interest which is closest to the exact division by the actuarial method. In the true amortization schedule the interest paid column decreases over time while the principal repaid column increases. The direct ratio method reflects this pattern while none of the other approximate methods do. In general, the direct ratio method will produce more accurate results than the other approximate methods.

The direct ratio method can best be illustrated by example. Consider a one-year loan repaid with 12 monthly installments. The sum of the positive integers from 1 through 12 is 78. The direct ratio method assumes that the interest paid is 12/78 of the finance charge in the first month, 11/78 in the second month, . . . , 1/78 in the last month. This decreasing pattern of interest payments will produce a corresponding increasing pattern of principal repayments. The direct ratio method is often called the *rule of 78*, despite the fact that the number 78 is valid only for a term of 12 months.

Table 8.4 is a modified amortization schedule for this method. We define S_r to be the sum of the first r positive integers, i.e.

$$S_r = 1 + 2 + \cdots + r = \frac{r(r + 1)}{2} . \tag{8.15}$$

The reader should verify the entries in Table 8.4 and note the various relationships in the table.

Formula (8.16) produces the following analogous formula for starting values using Chapter 3 notation

$$i \doteq \frac{2(n-k)}{k(n+1)+\frac{1}{3}(n-k)(n-1)}. \tag{8.17}$$

The derivation of formula (8.17) is left as an exercise. In general, formula (8.17) will produce results superior to those produced by formula (3.29).

In Section 8.2 calculation of the unearned finance charge on loans which are repaid early was discussed. There are two methods in widespread use. The first is based on an exact application of the actuarial method. The second is based on the rule of 78. It will be shown in Example 8.8 that the use of the rule of 78 favors the lender. The reader should be careful not to confuse this application of the rule of 78 with the derivation of formula (8.16). Two entirely different situations are involved.

***Example 8.6** Find the rate of interest on the installment loan in Example 8.1: (1) by the maximum yield method, (2) by the minimum yield method, (3) by the constant ratio method, and (4) by the direct ratio method (rule of 78).*

1. Using formula (8.11)

$$i^{max} = \frac{(2)(12)(80)}{(1000)(13)-(80)(11)} = .1584, \text{ or } 15.84\%.$$

2. Using formula (8.12)

$$i^{min} = \frac{(2)(12)(80)}{(1000)(13)+(80)(11)} = .1383, \text{ or } 13.83\%.$$

3. Using formula (8.13)

$$i^{cr} = \frac{(2)(12)(80)}{(1000)(13)} = .1477, \text{ or } 14.77\%.$$

4. Using formula (8.16)

$$i^{dr} = \frac{(2)(12)(80)}{(1000)(13)+\frac{1}{3}(80)(11)} = .1444, \text{ or } 14.44\%.$$

The exact answer produced by the actuarial method was found to be 14.45% in Example 8.1. The direct ratio method produces an answer closer to that obtained by the actuarial method than do any of the other approximate methods; and, in fact, it achieves a remarkable level of accuracy.

***Example 8.7** Compare the accuracy of formulas (3.29), (3.32), and (8.17) in obtaining starting values to solve for i, if $a_{\overline{20}|i} = 8.5136$.*

It can be shown in advance that $i = 10\%$. Using a contrived number will make it easier to compare the effectiveness of the three formulas.

Formula (3.29) gives

$$i_0 = \frac{2(n-k)}{k(n+1)} = \frac{2(20-8.5136)}{8.5136(21)} = .1285, \text{ or } 12.85\%.$$

Formula (3.32) gives

$$i_0 = \frac{1-\left[\frac{k}{n}\right]^2}{k} = \frac{1-\left[\frac{8.5136}{20}\right]^2}{8.5136} = .0962, \text{ or } 9.62\%.$$

Formula (8.17) gives

$$i_0 = \frac{2(n-k)}{k(n+1)+\frac{1}{3}(n-k)(n-1)} = \frac{2(20-8.5136)}{8.5136(21)+\frac{1}{3}(20-8.5136)(19)}$$
$$= .0913, \text{ or } 9.13\%.$$

Thus our highly-touted direct ratio (rule of 78) formula (8.17) beats the constant ratio formula (3.29) as expected, but loses out to the obscure formula (3.32).

However, if we apply formula (3.32) to the loan considered in Examples 8.1 and 8.6, we obtain

$$i = 12\,\frac{1-\left[\frac{11.1111}{12}\right]^2}{11.1111} = .1541, \text{ or } 15.41\%.$$

This result is considerably inferior to that produced by the direct ratio method. Thus, no one approximate formula will always produce the best starting values to use in solving for unknown rates of interest on annuities.

Experience has shown, however, that generally the direct ratio formula (8.17) will provide the best results for "shorter" terms, while formula (3.32) works best for "longer" terms. This difference is attributable to the fact that formula (3.32) more accurately reflects the compounding of interest over longer time periods.

***Example 8.8** Compute the unearned finance charge recovered by the borrower for the consumer loan in Example 8.1 if the loan is repaid in full after six regular installments are made: (1) using the actuarial method, and (2) using the rule of 78.*

The outstanding loan balance at the end of six months, prospectively, is

$$90\,a_{\overline{6}|.012043} = \$517.95.$$

The total that would have been paid over the final six months on the original payment schedule is

$$6 \cdot 90 = \$540.00.$$

Thus, the unearned finance charge is

time must equal the initial value of the asset less the amount in the sinking fund, i.e.

$$B_t = A - \left[\frac{A-S}{s_{\overline{n}|j}}\right] s_{\overline{t}|j}. \tag{8.21}$$

Note that formula (8.21) produces the proper values for B_0 and B_n. Then the depreciation charge is

$$\begin{aligned} D_t &= B_{t-1} - B_t \\ &= \left[A - \left[\frac{A-S}{s_{\overline{n}|j}}\right] s_{\overline{t-1}|j}\right] - \left[A - \left[\frac{A-S}{s_{\overline{n}|j}}\right] s_{\overline{t}|j}\right] \\ &= \left[\frac{A-S}{s_{\overline{n}|j}}\right] \left(s_{\overline{t}|j} - s_{\overline{t-1}|j}\right) \\ &= \left[\frac{A-S}{s_{\overline{n}|j}}\right] (1+j)^{t-1} \end{aligned} \tag{8.22}$$

It is evident from formula (8.22) that the sinking fund method produces depreciation charges which increase over the life of the asset. This may or may not be a reasonable pattern of depreciation charges, depending upon the nature of the asset. For example, it may produce a reasonable pattern of depreciation charges for an office building which depreciates slowly at first and then more rapidly later on. It probably would not produce a reasonable pattern of depreciation charges for an automobile.

However, it should be stressed that the choice of depreciation method often is not motivated by an analysis of which method produces the most realistic book values for the asset. A more important consideration for many individuals and firms is to use an allowable method that is most favorable from a tax perspective. An individual or firm that wants to have high tax deductions in the early years of the life of the asset would not select the sinking fund method.

It should not be thought that the sinking fund method actually requires the use of a sinking fund. It is merely a method of calculating book values and depreciation charges. A sinking fund may or may not actually be accumulated to replace the loss of capital.

The second, and simplest, method is the *straight line method*. This method is widely used in practice because of its simplicity. In this method the depreciation charge is constant, so that

$$D_t = \frac{A-S}{n}. \tag{8.23}$$

As a result the book values are linear

$$B_t = \left[1 - \frac{t}{n}\right] A + \frac{t}{n} S. \tag{8.24}$$

Note that formula (8.24) produces the proper values for B_0 and B_n. It is interesting to note that the straight-line method is a special case of the sinking-fund method in which $j = 0$.

The third method is the *declining balance method*, the *constant percentage method*, or the *compound discount method*. This method produces depreciation charges which decrease throughout the life of the asset, as opposed to the sinking fund method in which they increase and the straight line method in which they are constant.

The declining balance method is characterized by the fact that the depreciation charge is a constant percentage of the book value at the beginning of the period, i.e.

$$D_t = d \cdot B_{t-1}. \tag{8.25}$$

Now $D_t = B_{t-1} - B_t$ so that

$$B_t = B_{t-1}(1 - d)$$

and since this is true for all t we have

$$\begin{aligned}
B_0 &= A \\
B_1 &= B_0(1 - d) &&= A(1 - d) \\
B_2 &= B_1(1 - d) &&= A(1 - d)^2 \\
B_3 &= B_2(1 - d) &&= A(1 - d)^3 \\
&\vdots &&\vdots \\
B_t &= B_{t-1}(1 - d) &&= A(1 - d)^t \\
&\vdots &&\vdots \\
B_n &= B_{n-1}(1 - d) &&= A(1 - d)^n = S.
\end{aligned} \tag{8.26}$$

Since A and S are given, d can be found by

$$\begin{aligned}
A(1 - d)^n &= S \\
(1 - d)^n &= \frac{S}{A} \\
1 - d &= \left[\frac{S}{A}\right]^{\frac{1}{n}}
\end{aligned}$$

2. Using formulas (8.23) and (8.24) we have

t	D_t	B_t
0		10,000
1	1,800	8,200
2	1,800	6,400
3	1,800	4,600
4	1,800	2,800
5	1,800	1,000

3. Using formula (8.27) we obtain

$$d = 1 - (.1)^{.2} = 1 - .631 = .369$$

and using formulas (8.25) and (8.26) we have

t	D_t	B_t
0		10,000
1	3,690	6,310
2	2,328	3,982
3	1,470	2,512
4	927	1,585
5	585	1,000

4. Using formulas (8.29) and (8.30) we have

t	D_t	B_t
0		10,000
1	3,000	7,000
2	2,400	4,600
3	1,800	2,800
4	1,200	1,600
5	600	1,000

Example 8.10 Rework Example 8.9 using the 200% declining balance method.

Using formula (8.28), we have

$$d' = \frac{2}{5} = .4.$$

This is easily confirmed by general reasoning. Under the straight line method over five years the depreciation factor would be 20% per year (ignoring salvage value). The 200% declining balance factor must be twice that amount, i.e. 40%.

Using formula (8.25) and (8.26) we have

t	D_t	B_t
0		10,000
1	4,000	6,000
2	2,400	3,600
3	1,440	2,160
4	864	1,296
5	296	1,000

Note that $D_5 = 296$ is an arbitrary amount needed to make $B_5 = 1000$ exactly. If S had been ignored, D_5 would have been computed as $.4(1296) = 518.40$.

8.6 CAPITALIZED COST

This section discusses another type of analysis used in connection with fixed assets. An issue of considerable importance in practice is the comparison of the costs of alternate possible fixed assets. There are three costs involved in owning a fixed asset:

1. Loss of interest on the original purchase price, since that money could have been invested elsewhere at interest.
2. Depreciation expense.
3. Maintenance expense.

The *periodic charge* of an asset is defined to be the cost per period of owning the asset. If we let H be the periodic charge and M be the periodic maintenance expense, then the fundamental equation of value is

$$H = Ai + \frac{A - S}{s_{\overline{n}|j}} + M. \tag{8.31}$$

Formula (8.31) is quite analogous to formula (8.19). The term Ai is the loss of interest on the original purchase price. The term $\frac{A - S}{s_{\overline{n}|j}}$ is the periodic depreciation expense. The term M is the periodic maintenance expense. Recall that in formula (8.19) the term M is not needed, since R is net of expenses.

The *capitalized cost* of an asset is defined to be the present value of the periodic charges forever, i.e. the present value of a perpetuity for the amount of the periodic charge. The capitalized cost can be looked upon as the present value of maintaining an identical asset in operation indefinitely. Denoting the capitalized cost by K, we have

return it to the second party. The process of buying back the security is often called "covering the short."

The calculation of yield rates for short sale transactions presents some unanticipated difficulties. Consider the situation in which an investor sells a stock short for \$1000 and buys it back for \$800 at the end of one year. Clearly, a \$200 profit has been made, but what is the yield rate?

An equation of value might appear to be

$$1000(1 + i) = 800.$$

However, this equation produces a yield rate of $i = -20\%$ which is clearly unreasonable since a profit has been made.

It might be tempting to try to reverse the transaction and solve the equation of value

$$800(1 + i) = 1000$$

which at least gives a positive answer, i.e. $i = 25\%$. However, this answer cannot be justified either. It arises as a \$200 profit on an \$800 investment, but there never was an \$800 investment.

If, in fact, the transaction occurs exactly as stated, the yield rate does not exist. Some might prefer to say that the yield rate is infinite, since a profit was made on no investment. This point was illustrated previously in Example 5.3.

In practice, short sales normally do not occur as just illustrated. Governmental regulations in the United States required the short seller to make a deposit of a percentage of the price, e.g. 50%, at the time the short sale is made. This deposit is called the *margin* and cannot be recovered by the short seller until the short position is covered. The required margin percentage may be changed from time to time by the government. The reader should note in this context the word "margin" has a totally different meaning than it had in Section 8.3 in connection with adjustable rate mortgages.

Thus, in the above illustration the short seller would have to deposit margin of \$500 at the time of the short sale if the margin requirement is 50%. Now the situation is such that a valid yield rate can be computed. A \$200 profit is made and \$500 was deposited over a one-year period of time, so that the yield rate is 40%.

Actually the situation is slightly more complicated than just described. The short seller will be credited with interest on the margin deposit which will increase the yield rate somewhat. If the short seller is credited with 8% interest on the margin deposit, then the amount of interest earned will be .08(500) = 40. The yield rate taking this interest into account becomes 240/500 = 48%.

The astute reader may ask about interest on the proceeds of the original short sale. In the above illustration could the short seller also earn interest on

the $1000 proceeds from the short sale? The answer is no. Governmental regulations require that these proceeds remain in a non-interest bearing special account until the short position is covered at which time these funds will be used for the purchase necessary to cover the short position. Any positive residual is the profit on the transaction, while any negative residual is the loss on the transaction.

In practice, if the short position develops a loss, additional margin may be required prior to the position being covered. Conversely, if the position develops a profit, some of the margin may be released and can be withdrawn or used for other purposes. These types of adjustments are controlled by governmental regulation.

One other aspect of short selling is significant. If the security in question pays dividends (e.g. on stocks) or interest (e.g. on bonds), then the short seller is required to pay these to the purchaser of the security. This will serve to decrease the yield rate realized.

For example, in the above illustration, if the stock sold short pays $60 in dividends during the year, the short seller's net profit is as follows:

Gain on short sale	+ 200
Interest on margin	+ 40
Dividends on stock	- 60
Net profit	+ 180

Now the yield rate becomes 180/500 = 36%. This requirement creates a significant disincentive to sell short any securities which pay a significant amount of dividends or interest.

Short selling by itself is a speculative endeavor and should not be entered into lightly. The short seller is counting on a significant decline in the value of the security in order to come out ahead. Speculative short selling is normally done over relatively brief periods, so that it is unusual for short positions to remain open for extended periods of time.

However, investment strategies have been developed that involve combinations of long and short positions in related securities which have greatly reduced risk and also have an excellent prospect to earn a reasonable return on investment. A generic name for these types of transactions is ***hedging***, and many of the transactions become quite complex. In fact, occasionally situations of this type arise in which a profit is certain. Such transactions of guaranteed profitability are called *arbitrage*. Generally, arbitrage opportunities are fleeting since market prices react quickly to eliminate such golden opportunities.

Mutual funds

Mutual funds are pooled investment accounts in which the individual investor buys shares. Their primary advantage is to offer a greater degree of diversification than individuals could achieve on their own.

Originally, mutual funds were primarily invested in common stocks. More recently, mutual funds offering a broad array of other investment options have appeared. Even within the traditional category of common stock mutual funds considerable variation exists among various funds in their relative emphases on safety of principal, income, and growth.

Mortgage backed securities

Mortgage backed securities (MBS) are securities which are created out of a defined pool of real estate mortgages. They are issued by three governmentally owned or chartered corporations holding large numbers of mortgages. Investors receive periodic payments consisting of both principal and interest. The rate at which principal is repaid is variable depending on the rate at which the underlying mortgages are paid off. The three corporations are as follows:

1. The Government National Mortgage Association (GNMA) issues MBS called "Ginnie Maes." GNMA is owned by the U. S. Government and payment of principal and interest is guaranteed by the full faith and credit of the U. S. Government.

2. The Federal Home Loan Mortgage Corporation (FHLMC) issues MBS called "Freddie Macs." FHLMC is chartered by the U. S. Government and the corporation (but not the government) guarantees payment of principal and interest.

3. The Federal National Mortgage Association (FNMA) issues MBS called "Fannie Maes." FNMA is chartered by the U. S. Government and the corporation (but not the government) guarantees payment of principal and interest.

Collateralized mortgage obligations

Collateralized mortgage obligations (CMOs) are a newer type of financial instrument designed to improve upon the traditional MBS. CMOs involve the same type of investment in real estate mortgages as MBS, but involve complex structuring of the portfolios designed to lessen the uncertainty in cash flow that is typical in traditional MBS.

However, they do not have a specific maturity date and are sold on a basis hich quotes their "average life." The average life is the average amount of me that each dollar of principal is expected to be outstanding, assuming a asonable prepayment schedule for the mortgages in the pool. CMOs are ructured with a wide range in average life in order to provide more choice to e investor.

CMOs typically offer higher yields than corporate bonds in order to mpensate for the uncertainly of their repayment schedule and the attendant investment risk. CMOs make payments monthly or quarterly rather than miannually as with bonds. CMOs are quite liquid, since there is an active arket in existing issues. Prices in the market for existing CMOs vary versely with prevailing interest rates, similarly to bond prices.

ptions

Options are financial instruments which give the owner the right to buy or ll a security on a future date at a fixed price, called the *exercise price* or *iking price*. There are two types of options: calls and puts. A *call* gives the ner the right to purchase the security at the exercise price, while a *put* gives owner the right to sell the security at the exercise price. With European tions the exercise of the option must occur on a fixed date at which time the tion expires. With American options the exercise can occur on any future te through the expiry date.

Investors can either buy or sell options. Thus, if an investor thinks the urity price is likely to rise, then a call should be purchased or a put should sold. Conversely, if an investor thinks the security price is likely to fall, n a put should be purchased or a call should be sold.

There are two primary motivations for buying or selling options, which are pposite ends of the spectrum. One motivation is speculation. Option prices highly volatile, which leads to a large potential for profit and a large risk loss. For example, consider a security selling at \$50 with a call option to the security at \$45. The option is worth at least \$5 (it will sell for more n \$5 prior to expiry). If the security increases in price to \$55 the option is rth at least \$10. Thus, a 10% increase in the price of the underlying security ds approximately to a 100% increase in the value of the option. This effect alled *leverage*.

The second motivation is quite the opposite. Options offer great flexibility developing hedging strategies to reduce investment risk. Hedging was scussed briefly in Section 8.7. Many of the option hedging investment ategies have become quite complex, and require computer analysis to plement effectively.

3. A loan of \$12,000 is to be repaid in one year according to one of the following arrangements:
   A - \$1000 payable at the end of each month in addition to a finance charge of \$1000 payable when the loan is approved.
   B - Repayment at the end of each month according to an amortization schedule with $i^{(12)} = 12\%$.
   Find the difference in the amount of interest paid under options A and B.

4. Find the APR for both options A and B in Exercise 3.

5. A borrower visits three banks to obtain quotes on a car loan repayable with 24 monthly installments. The first bank quotes a monthly payment of X based on a total carrying charge equal to the product of the initial balance, the number of years to repay, and 6.5%. The second bank quotes a monthly payment of Y based on an annual effective interest rate of 12.6%. The third bank quotes a monthly payment of Z based on an interest rate of 12% covertible monthly. Rank the values of X, Y and Z.

6. A loan of \$8000 at an interest rate of 12% per annum is to be repaid with three payments:
   - A payment of \$2000 at the end of 3 months.
   - A payment of \$4000 at the end of 9 months.
   - A payment of \$$X$ at the end of 12 months.

   Find X:
   a) Using the United States Rule.
   b) Using the Merchant's Rule.

7. A borrows \$10,000 for two years at an effective rate of interest of 10%. A agrees to pay interest at the end of each year and to repay the principal at the end of two years. At the end of one year A is able to pay only \$500. Find the amount necessary to completely repay the loan at the end of two years:
   a) Using the actuarial method.
   b) Using the United States Rule.

8. A borrower deposits \$200 immediately for the guarantee to be able to borrow \$1000 at the end of one year. The borrower must repay the \$1000 at the end of two years.
   a) Find the two positive yield rates for this transaction.
   b) Truth in lending does not resolve this situation since there is only one advance. Compute the APR using the method of equated time on the payments rather than on the advances.

9. The tables promulgated under truth in lending for handling irregular transactions define three terms as follows:
   A - number of regular payments.
   B - equivalent single payment point (exact, not by the method of equated time).
   C - finance charge per \$1000 of payment (related to payments by borrower, not to advances).

Show that the following series of identities is valid:

$$A - a_{\overline{n}|} = A - A(1 + i)^{-B} = C\left[\frac{A}{1000}\right].$$

8.3 Real estate mortgages

10. A family purchases a house for \$160,000. They agree to put 25% down and finance the balance with a 30-year fixed rate mortgage at 9%. In order to secure this loan they must pay 2 points at settlement. They go to settlement on September 16. Find the total amount of interest paid during the first calendar year of the mortgage, if 1 1/2 points is deemed to be interest and the family makes payments on the first day of each month.

11. Find the APR on the mortgage loan in Exercise 10, assuming that the other 1/2 point and all other closing costs do not have to be reflected in the APR.

12. A 15-year mortgage has monthly payments of \$1000 with interest convertible monthly. At the end of each month, the borrower makes a \$1000 payment. In addition to the regular monthly payment, the borrower makes an additional payment equal to the amount of principal that would have been repaid in the next regular monthly payment. Under this arrangement the loan will be completely repaid after 90 payments. Show that the amount of interest saved over the life of the loan is equal to

$$90{,}000 - 1000\frac{\ddot{a}_{\overline{180}|}}{s_{\overline{2}|}}$$

where the annuity symbols are computed at the monthly rate of interest on the loan.

13. A builder took out a \$2,000,000 construction loan disbursed in three installments. The first installment of \$1,000,000 is disbursed immediately and this is followed by two \$500,000 installments at six month intervals. The interest on the loan is calculated at a rate of 15% convertible semiannually and accumulated to the end of the second year. At that time, the loan and accumulated interest will be replaced by a 30-year mortgage at 12% convertible monthly. The amount of the monthly mortgage payment for the first five years will be one-half of the payment for the sixth and later years. The first monthly mortgage payment is due exactly two years after the initial disbursement of the construction loan. Find the amount of the 12th mortgage payment. Answer to the nearest dollar.

14. A \$100,000 loan is to be repaid by 30 equal payments at the end of each year. Interest on the loan is at 8% effective. In addition to the annual payments, the borrower must pay an origination fee at the time the loan is made. This fee is 2% of the loan but does not reduce the loan balance. When the second payment is due, the borrower pays the remaining loan balance in full. Determine the yield to the lender considering the origination fee and the early pay-off of the loan.

32. A piece of equipment that was purchased for \$15,000 will have a salvage value of \$2000 after 15 years. Its book value has been determined by depreciation in accordance with the compound interest method, using an interest rate of 5% per annum. At the end of the 10th year, the depreciation method is changed to the straight line method for the remaining 5 years. Determine the book value at the end of the 12th year. Answer to the nearest dollar.

33. A company buys two machines. Both machines are expected to last 14 years, and each has a salvage value of \$1050. Machine A costs \$2450, while Machine B costs \$$Y$. The depreciation method used for Machine A is the straight line method, while the depreciation method used for Machine B is the sum of the years digits method. The present value of the depreciation charges made at the end of each year for Machines A and B are equal. If the effective rate of interest is 10%, calculate Y.

34. A new machine which costs \$11,000 has a salvage value of \$900. The machine has a useful lifetime of 100 years. Let:
    $BVSL_t$ = the book value of the asset at the end of year t under the straight line depreciation method.
    $BVSD_t$ = the book value of the asset at the end of year t under the sum of the years digits depreciation method.
    For what value of t is $BVSL_t - BVSD_t$ a maximum?

35. Assume that book values of an asset are continuous functions of t. Find an expression for that value of t at which the excess of B_t by the straight line method over B_t by the constant percentage method is a maximum.

36. An asset costing \$5000 has a salvage value of \$2000 at the end of n years. If the depreciation charge for the 12th year is \$100 using the sum of the years digits method, find n.

8.6 Capitalized cost

37. A machine sells for \$10,000 and has a salvage value of \$1000 at the end of 10 years. The annual maintenance expense of the machine is \$500. Assuming 5% interest:
    a) Calculate the periodic charge of the asset.
    b) Calculate the capitalized cost of the asset.

38. Machine 1 sells of \$1000 with a salvage value of \$50 at the end of nine years. Machine 2 sells for \$1100 with a salvage value of \$200 at the end of nine years. At what rate of interest would a purchaser be indifferent between the two machines? Assume equal maintenance expenses for the two machines.

39. Plastic trays last 8 years and cost \$20 each. Metal trays last 24 years and cost \$$X$ each. Trays are needed for 48 years, and inflation will increase the cost of the trays 5% per year. At 10.25% interest, determine X so that the buyer is indifferent between purchasing plastic or metal trays.

40. Rework Example 8.12 assuming that the cost of poles is expected to increase by 2% per year indefinitely into the future.

41. A construction firm buys \$1000 worth of lumber. Find the maximum amount the firm would be willing to pay to treat the lumber to extend its life from 10 to 15 years if the salvage value in either case is \$50. The effective rate of interest is 3 1/2%.

42. Machine 1 sells for \$100,000, has an annual maintenance expense for the first year of \$3000, and has a useful life of 20 years with no salvage value. Machine 2 has an annual maintenance expense for the first year of \$10,000 and has a useful life of 15 years with no salvage value. It is anticipated that the cost of the machines and the annual maintenance expenses will increase by 4% per year indefinitely into the future. Machine 2 produces output twice as fast as Machine 1. Maintenance expenses are paid at the beginning of each year. Assuming the effective rate of interest is 8%, find the price of Machine 2 at which a buyer would be indifferent between the two machines. Answer to the nearest \$100.

8.7 Short sales

43. In the illustration in this section the yield rate was 36% when considering both the interest on margin and the payment of dividends. Find the yield rate if the margin requirement had been 60% instead of 50%.

44. Pursuing Exercise 43 one step further, if the margin requirement is m, where $0 < m \leq 1$, find a general formula for the yield rate as a function of m.

On the demand side, the primary issue is the *productivity of capital*. Virtually all business firms need capital on which to operate successfully. Some of this capital generally comes from borrowing. In the long run, the firm will be successful only if the return on capital employed is greater than the cost of borrowing. Of course, not all borrowing is done by firms, but much is also done by individuals and the government. Admittedly, a portion of this borrowing is used to finance current consumption. However, a portion of it is also in some sense being invested, e.g. individuals borrowing to purchase homes and to finance college educations, and the government borrowing to build infrastructure such as roads and airports. In the end, a healthy economy will require that a substantial portion of money being borrowed is invested productively.

Although these two major theories are quire different, they are in no way incompatible. In fact, quite the contrary, they serve to reinforce each other.

The above discussion barely scratches the surface of some of the economic or even psychological and philosophical, theories attempting to explain the existence of interest. However, the above discussion is sufficient for the purposes of this book. Curious readers who would like to explore this area more completely are encouraged to refer to relevant literature in economics and finance.

9.3 DETERMINANTS OF THE LEVEL OF INTEREST RATES

Section 9.2 provides a rationale for the existence of interest. However there was little in that analysis that sheds light on the level of interest rates that exist at various points in time.

If we examine interest rates existing in the past, we discover that great variation has occurred over time. In 1945 the average yield on Treasury bills was 0.33%. In 1981 the same security had an average yield of 14.71%. In August 1980 the prime rate (to be defined later in this section) was 11%. In December 1980 it was 21.5%. What can explain variations of this magnitude?

In looking at the level of interest rates it is necessary to distinguish between two types of variation. The first type of variation is general and pertains to the overall level of interest rates prevailing in the market. The second type of variation is specific and pertains to the level of the interest rate on a particular transaction.

Basic economic theory would suggest that rates of interest, like other prices, are established by supply and demand. If the demand for funds to borrow is strong in relation to the availability of funds, interest rates will rise. Conversely, if the demand for funds to borrow is weak in relation to the availability of funds, interest rates will fall. It sounds simple, but in practice

there are a large number of factors that come together in complex ways to determine rates of interest.

The following is a list of major factors which have an influence on the level of the rate of interest. The list is not exhaustive, but it does include most of the major determinants.

1. *The underlying "pure" rate of interest*
   Most economic and financial theorists believe that there is an underlying "pure" rate of interest as a base which is related to long-term productivity growth in the economy. This rate would prevail on a risk-free investment if there were no inflation. This rate has proven to be relatively stable over many decades. In the United States this rate has typically been in the range of 2% to 3%.
2. *Inflation*
   Experience has shown that inflation has a significant effect on the rate of interest. This factor is discussed in more detail in Section 9.4.
3. *Risk and uncertainty*
   Experience has also shown that risk and uncertainty have a significant effect of the rate of interest. This factor is discussed in more detail in Section 9.5.
4. *Length of investment*
   There will normally be differences in the market between the rates of interest on short-term and long-term loans and investments, all other things being equal. This phenomenon is discussed in more detail in Section 9.6.
5. *Quality of information*
   In finance theory "efficient" markets are defined as those in which all buyers and sellers (in this context borrowers and lenders) possess the same information. Aberrations in the rate of interest are more likely to exist in "inefficient" markets. In the modern computer-information age, markets tend to be more efficient than in the past. However, certain market rigidities remain which can affect the rate of interest.
6. *Legal restrictions*
   Some rates of interest are regulated by the government. In the United States there has been a trend toward deregulation in recent years, so that this has become a less significant factor than in the past. Nevertheless some rates of interest still are subject to some degree of regulation.
7. *Governmental policy*
   The federal government has a major influence, even control, on the overall level of interest rates through its monetary and fiscal policy. The primary control is the ability of the Federal Reserve Board to adjust the supply of

$$1 + i' = \frac{1+i}{1+r} \tag{9.3a}$$

$$i' = \frac{i-r}{1+r}. \tag{9.3b}$$

According to the theory outlined above, i' will be relatively stable over time. However, i and r will tend to move up and down together. The above relationships are not precise. The real rate i' is not totally constant over time, nor is the correlation between i and r exact either. Also, we have the dual problems of measurement and time lags, since r should be the *expected* rate of inflation. Thus, these formulas should not be thought of as exact relationships, but rather as convenient rules of thumb.

Nevertheless, formula (9.3*a*) is quite useful in performing calculations involving rates of inflation. For example, assume that we wish to find the present value of a series of payments at the end of each period for n periods in which the base payment amount at time 0 is R, but each payment is indexed to reflect inflation. If r is the periodic rate of inflation and i is the periodic rate of interest, then the present value of this series of payments is

$$R\left[\frac{1+r}{1+i} + \frac{(1+r)^2}{(1+i)^2} + \cdots + \frac{(1+r)^n}{(1+i)^n}\right] = R(1+r)\frac{1-\left[\frac{1+r}{1+i}\right]^n}{i-r} \tag{9.4}$$

by applying formula (4.39).

However, if we use formula (9.3*a*) the formulation of the problem becomes

$$R\left[\frac{1}{1+i'} + \frac{1}{(1+i')^2} + \cdots + \frac{1}{(1+i')^n}\right] = R\,a_{\overline{n}|i'}. \tag{9.5}$$

Formulas (9.4) and (9.5) produce the same numerical answer and have an interesting verbal interpretation. Formula (9.4) represents the present value of the payments *including* inflation computed at the *nominal* rate of interest. However, in formula (9.5) this is seen to equal the present value of the payments *excluding* inflation computed at the *real* rate of interest.

The above verbal interpretation really provides more general guidance in computing present values of future payments in practical situations as follows:

1. If future payments are not affected by inflation, then discount at the *nominal* rate of interest.
2. If future payments are adjusted to reflect the rate of inflation and the adjustment is reflected in the payment amount, then also discount at the *nominal* rate of interest.

3. If future payments are adjusted to reflect the rate of inflation but the adjustment is not reflected in the payment amount, the correct procedure is to discount at the *real* rate of interest.

The above discussion involves a present value analysis. It is also instructive to consider inflation in connection with accumulated values. Consider the common situation in which an investor invests A dollars for n periods at interest rate i. The value of this investment in "nominal dollars" at the end of n periods is

$$A(1 + i)^n . \tag{9.6}$$

However, how much is this investment really worth at that future date? If the rate of inflation is r, then the purchasing power of this investment at the end of n periods is

$$A \frac{(1 + i)^n}{(1 + r)^n} = A(1 + i')^n . \tag{9.7}$$

Thus, the value of this investment in "real dollars" is lower, since $i > i'$. Many investors would find it quite enlightening to analyze their investment programs in terms of "real" results as well as "nominal" results, rather than considering only the latter.

In the above discussion, there has been the implicit assumption that $i > r$, i.e. that the nominal rate of interest is greater than the rate of inflation. In general, this relationship will hold, particularly over significant periods of time.

However, in some cases, for at least some investors, this may not be the case. Such a result is most likely to occur during periods of high inflation. For example, in the United States during 1979-1981 the rate of inflation, as measured by the Consumer Price Index (CPI), which is probably the most widely used index of inflation, was in "double digits" (i.e. over 10%). Yet during this same period of time billions of dollars were invested in savings accounts at rates of interest in the 7% range. For all those investors, the "nominal value" of their savings was increasing, but the "real value" was declining.

Readers who are interested in pursuing a classical analysis of the relationship between rates of interest and rates of inflation are referred to the book by I. Fisher (1930) listed in the bibliography.

Example 9.1 An insurance company is making annual payments under the settlement provisions of a personal injury lawsuit. A payment of $24,000 has just been made and ten more payments are due. Future payments are indexed to the Consumer Price Index which is assumed to increase at 5% per

interest rate. In general, the greater the risk in an investment the higher the risk premium.

This computed yield rate is somewhat misleading, however. After the fact, for this one bond the yield rate will actually turn out to be 14.89% if no default occurs. However, if total default occurs, the actual realized yield rate will be −100%. If partial default occurs, the realized yield rate will be somewhere in between.

We might be interested in knowing the probability of default which is implicit in the purchase of this high-risk bond. We define the *expected present value* (EPV) of a future payment as its present value multiplied by the probability of payment. We can compute the implicit probability of payment, denoted by p, as

$$940 = p\frac{1080}{1.08}$$

which gives $p = .94$. Thus, we have an implicit probability of default equal to .06. Note that the present value is computed at the risk-free rate of 8%.

The above analysis is valid as far as it goes, but it needs refinement. It is quite unlikely that bond purchasers would be willing to pay \$940 for this bond if they really thought that the probability of default is as high as .06. Why would investors buy a high-risk investment with an expected yield rate of only 8% when they could buy a risk-free investment with the same yield rate?

Thus, a more reasonable interpretation of a price like \$940 for such a high-risk bond is that the \$60 price differential partially represents the probability of default and partially represents a higher return to the purchaser as compensation for the assumption of risk.

Let us assume that investors think that assumption of this level of risk is worth an extra 3% in the yield rate, i.e. 11% instead of 8%. Then, the implicit probability of payment p can be determined from

$$940 = p\frac{1080}{1.11}$$

which gives $p = .9661$. Thus, the implicit probability of default is .0339. It is obvious that this answer is not unique and that other combinations of yield rates and probabilities of default would also produce a price equal to \$940.

The above analysis considers one bond bought in isolation. Now consider a diversified portfolio of bonds all of which are similar to the above bond, so that the law of large numbers applies. If the actual rate of default on the portfolio turns out to be .0339, then the yield rate on the entire portfolio will be 11%. Of course, in practice the rate of default will not be exactly .0339 but will follow a probability distribution of some type. The results of mathematical

statistics can then be applied to make probability statements about various possible yield rates to be expected on the overall portfolio. This will be illustrated in Example 9.3.

Now, let us turn this illustration around and look at it another way. Assume we want to quantify the risk factor separately from the interest rate. Thus, if the market price of the bond is \$940 and the risk-free rate of interest is 8%, then the value at the end of the year would be $940(1.08) = \$1015.20$. This way of quantifying risk adjusts the expected payoff downward from \$1080 to \$1015.20 to reflect risk, but then computes present values at the risk-free rate of interest.

This second approach illustrates an error that is sometimes made in financial analysis. One way of quantifying risk is to adjust the interest rate. A second way of quantifying risk is to adjust the payment amount. These are two different, but both valid, ways of quantifying risk. The mistake that is sometimes made is to do both, i.e. lower the expected payoff and raise the interest rate. This procedure is flawed, since it "double counts" the risk involved.

We now generalize these results to more complex situations involving multiple payments. Consider a series of future payments $R_1, R_2, \ldots, R_n$ that are made at times $1, 2, \ldots, n$. For example, on a 10-year \$1000 bond with 8% annual coupons we have $R_1 = R_2 = \cdots = R_9 = 80$ and $R_{10} = 1080$. Assume that the probabilities of payment are $p_1, p_2, \ldots, p_n$, respectively. Then the expected present value of this series of payments is given by

$$EPV = \sum_{t=1}^{n} R_t(1+i)^{-t}p_t \tag{9.8}$$

where i is an appropriate rate of interest reflecting the risk involved as discussed above.

Formula (9.8) involves three key values: (1) the expected present value EPV, (2) the yield rate of interest i, and (3) a set of probabilities of payment p_t for $t = 1, 2, \ldots, n$. Situations arise in practice in which we know two of these values and wish to determine the third.

One obvious complication is that there are many patterns of probabilities p_t that might be used. One assumption that is often made in connection with default risk is to assume that the probability of default is constant during each period. Let this constant probability of default during one period be denoted by q. The corresponding probability of non-default is then $p = 1 - q$. Under these assumptions, the probability that the tth payment will be made is

$$p_t = p^t \tag{9.9}$$

One final observation is to note that we have not dealt with reinvestment rate risk in this section. This is a very important risk, but it is addressed elsewhere in this book.

***Example 9.2** The prevailing yield rate for risk-free 10-year bonds is 9% effective. A $1000 10-year bond with annual coupons is issued on which the coupon rate is 9%. (1) Find the price an investor would be willing to pay if the probability of default each year is .005 and if the investor requires a yield rate of 11% in compensation for the risk of default. (2) Find the risk premium in the interest rate in this transaction.*

1. If the bond were risk-free, it would obviously sell at par, i.e. the price would be $1000. We can utilize formula (9.10) to find the price

$$90\left[\frac{.995}{1.11}+\left(\frac{.995}{1.11}\right)^2+\cdots+\left(\frac{.995}{1.11}\right)^{10}\right]+1000\left(\frac{.995}{1.11}\right)^{10}=\$852.825$$

upon summing the geometric progression.

2. We have the following equation of value to determine the yield rate on the bond

$$852.825 = 90a_{\overline{10}|i} + 1000(1+i)^{-10}.$$

If we solve this equation by iteration using the approach developed in Section 7.6, we obtain the yield rate $i = 11.56\%$. Thus, the risk premium in the interest rate is $11.56\% - 9\% = 2.56\%$. It is instructive to note that the risk premium is approximately equal to the difference between the two interest rates, i.e. 11% minus 9%, plus the default rate, 0.5%.

***Example 9.3** Consider the illustration in this section in which a $1000 one-year 8% bond selling for $940 bought at a yield rate of 11% has an implicit probability of default equal to .0339. Assume that a diversified portfolio of 500 such bonds is purchased. Use the normal distribution to find a 95% confidence interval for the average probability of default on the entire portfolio.*

The standard deviation for the probability of default is equal to

$$\sigma_p = \sqrt{\frac{pq}{n}} = \sqrt{\frac{(.0339)(.9661)}{500}} = .0081.$$

Thus, the 95% confidence interval using the normal distribution is

$$\mu_p \pm 1.96\,\sigma_p = .0339 \pm 1.96(.0081)$$

or (.0180, .0498). When translated into yield rates, the confidence interval becomes (9.17%, 12.83%). Readers desiring a brief statistical review are referred to Appendix IV.

9.6 YIELD CURVES

Another factor that affects the level of interest rates is the length of the investment period. Typically, at any point in time short-term and long-term rates of interest are different. This phenomenon is called the *term structure of interest rates*. Table 9.2 is a hypothetical table of interest rates illustrating the term structure.

Table 9.2 Hypothetical Term Structure of Interest Rates

Length of investment	*Interest rate*
1 year	7.00%
2 years	8.00
3 years	8.75
4 years	9.25
5 years	9.50

Table 9.2 illustrates a situation in which the rate of interest increases as the length of the investment period increases. In other words, long-term interest rates are higher than short-term interest rates. This pattern is usually found in practice. Periods beyond five years are of concern as well, but Table 9.2 was kept short for simplicity in illustrating the concept.

Another phrase used to describe the term structure of interest rates is the *yield curve*. Figure 9.1 graphically illustrates the yield curve for the rates of interest contained in Table 9.2.

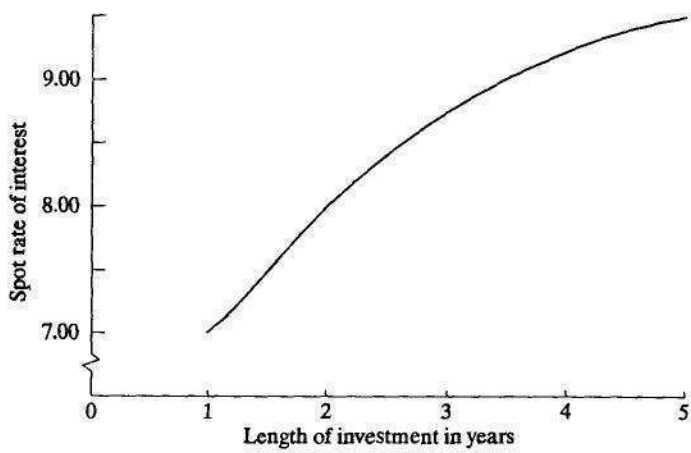

Figure 9.1 Illustrative yield curve

B and this will be reflected when the computation is based on different spot rates from the yield curve. An analytical approach to measuring this difference in average bond terms is given in Section 9.8.

Another type of interest rate is the *forward rate*. This is a spot rate that will come into play in future. In essence, a forward rate can be considered as a future reinvestment rate.

In order to illustrate this concept, consider a business firm which needs to borrow a sizable amount of money for two years. The firm is presented with the yield curve illustrated in Table 9.2. The firm has two options. The first option is to borrow for two years at the 2-year spot rate of 8%. The second option is to borrow for one year at the 1-year spot rate of 7% and then borrow for the second year at the 1-year spot rate in effect a year later. This 1-year spot rate for the second year is called a "forward rate."

We will now analyze the two options. Let the forward rate be denoted by f. The firm will then be indifferent between the two options if

$$(1.08)^2 = (1.07)(1 + f)$$

which is solved to give $f = .0901$, or 9.01%. Thus, if the firm expects the forward rate to be greater than 9.01%, it should use the first option to borrow. However, if the firm expects the forward rate to be less than 9.01%, it should use the second option. The reader should note that the symbol f was also used in Section 5.9 to represent the project financing rate.

Note that in referring to forward rates we need to be precise in specifying both the period of deferral and the period for which the rate applies. For example, a projected forward rate of interest for the interval between 3 and 8 years from the present is, in essence, an estimated 3-year deferred 5-year spot rate. The forward rate f used in the above illustration is an estimated 1-year deferred 1-year spot rate.

The use of spot rates and forward rates in the term structure of interest rates introduces a more sophisticated way of analyzing alternatives for investing and borrowing than we have considered previously. It has become the basis for many complex modern financial transactions, including extensive hedging and arbitrage strategies.

This section is not a complete treatment of the subject but it does introduce the basic concepts. The reader is encouraged to pursue the subject in more depth in the references listed in the bibliography.

***Example 9.4** Find the present value of payments of $1000 at the end of each year for five years using the spot rates given in Table 9.2. What level yield rate would produce an equivalent value?*

The present value of the payments is

$$1000[(1.07)^{-1} + (1.08)^{-2} + (1.0875)^{-3} + (1.0925)^{-4} + (1.095)^{-5}] = \$3906.63.$$

The equivalent level yield rate is found by solving $a_{\overline{5}|i} = 3.90663$. If we solve this by iteration using the techniques discussed in Section 3.8, we obtain $i = .0883$, or 8.83%.

***Example 9.5** Find the present value of the remaining payments in the annuity given in Example 9.4 immediately after two payments have been made. The forward rates at that time are expected to be 1% higher for all periods than the current spot rates.*

The comparison date for this calculation is at the end of two years. At that time there are three remaining annuity payments to be made. The expected forward rates are the current spot rates for 1, 2, and 3 years (not for 3, 4, and 5 years) each increased by 1%. Thus, the remaining present value is

$$1000[(1.08)^{-1} + (1.09)^{-2} + (1.0975)^{-3}] = \$2524.07.$$

9.7 INTEREST RATE ASSUMPTIONS

Within the realm of standard borrowing and lending transactions the rate of interest involved is typically specified or implied by the provisions of the transaction. For example, loan rates on mortgages and yield to maturity rates on bonds are either known or can readily be determined. The same can be said of other borrowing and lending transactions we have discussed in this book.

However, as we have also seen, the use of discounted cash flow analysis is a powerful analytical and decision-making tool with applicability going well beyond borrowing and lending transactions. In particular, the discussion of capital budgeting, net present value, and internal rate of return in Chapter 5 was motivated by the need to analyze more general business and financial transactions involving estimated future receipts and/or disbursements at various points of time.

This type of analysis is widely used in the field of actuarial science in which present value analysis of future contingent events is a fundamental activity. Another area in which this type of analysis is becoming increasingly important is in the field of accounting. An increasing number of values appearing on financial statements are based on present value calculations. Many other examples could also be cited illustrating the importance of discounted cash flow analysis in financial analysis and decision making.

In using discounted cash flow analysis for these types of more complex situations, the decision maker must resolve three basic ingredients listed in Section 9.5:

Let $R_1, R_2, \ldots, R_n$ be a series of payments made at times $1, 2, \ldots, n$. The method of equated time was defined in formula (2.9). If we apply this formula with the required change in notation, we obtain

$$\bar{t} = \frac{\sum_{t=1}^{n} t \cdot R_t}{\sum_{t=1}^{n} R_t}. \tag{9.12}$$

For example, consider the two bonds cited above. If the coupons are paid annually, then the average term to maturity of the 5% bond is

$$\bar{t} = \frac{1 \cdot 5 + 2 \cdot 5 + \cdots + 10 \cdot 5 + 10 \cdot 100}{5 + 5 + \cdots + 5 + 100} = 8.50.$$

Similarly, the average term to maturity of the 10% bond is

$$\bar{t} = \frac{1 \cdot 10 + 2 \cdot 10 + \cdots + 10 \cdot 10 + 10 \cdot 100}{10 + 10 + \cdots + 10 + 100} = 7.75.$$

The 5% bond has an average payment date of 8.50 years, while the 10% bond has an average payment date of 7.75 years. Thus, we could say that the 5% bond is a longer term bond than the 10% bond. This confirms the statement we made comparing Bonds A and B in Section 9.6.

An even better index is given by *duration*. The concept behind duration is similar to the method of equated time, except that the present value of each payment is used as the weight instead of the payment itself. Then duration, denoted by $\bar{d}$, is given by

$$\bar{d} = \frac{\sum_{t=1}^{n} t v^t R_t}{\sum_{t=1}^{n} v^t R_t}. \tag{9.13}$$

Note that $\bar{d}$ is a function of i.

Several special cases of formula (9.13) are of importance:

1. If $i = 0$, then $\bar{d} = \bar{t}$. This is obvious upon inspection, since formula (9.13) immediately simplifies to formula (9.12). Thus, the method of equated time is, in essence, a special case of duration which ignores interest.
2. The duration $\bar{d}$ is a decreasing function of i. The proof of this result will come later in this section. However, the result does have an interesting verbal interpretation. As the rate of interest increases, the terms in the numerator of formula (9.13) with the higher values of t are discounted

relatively more than the terms with the lower values of t. This serves to reduce the overall weighted average value of t, i.e. $\bar{d}$.

3. If there is only one future payment, then $\bar{d}$ is the point in time at which that payment is made. This intuitively appealing result is obvious upon examining formula (9.13), since the summations in the numerator and denominator have only one term each and everything cancels except the time of payment.

It is also instructive to examine the rate of change in the present value of a series of future payments as the rate of interest changes. Let this present value be denoted by $P(i)$, i.e.

$$P(i) = \sum_{t=1}^{n} v^t R_t = \sum_{t=1}^{n} (1+i)^{-t} R_t. \tag{9.14}$$

We now define the *volatility* of this present value, denoted by $\bar{v}$, as

$$\bar{v} = -\frac{P'(i)}{P(i)} \tag{9.15}$$

where $\bar{v}$ is a function of i.

The reasoning behind formula (9.15) is analogous to the reasoning behind the definition of the force of discount given in Section 1.9. The term $P'(i)$ measures the instantaneous rate of change in the present value of the payments as the rate of interest changes. Dividing by $P(i)$ expresses this instantaneous rate of change in units independent of the size of the present value itself. The minus sign is necessary to make $\bar{v}$ positive, since $P'(i)$ is negative.

The word "volatility" seems appropriate to describe $\bar{v}$, since $\bar{v}$ is a measure of how rapidly the present value of a series of future payments changes as the rate of interest changes.

We obtain a very interesting result if we substitute formula (9.14) into formula (9.15)

$$\begin{aligned}
\bar{v} &= -\frac{P'(i)}{P(i)} \\
&= -\frac{\dfrac{d}{di}\displaystyle\sum_{t=1}^{n}(1+i)^{-t}R_t}{\displaystyle\sum_{t=1}^{n}(1+i)^{-t}R_t} \\
&= \frac{\displaystyle\sum_{t=1}^{n} t(1+i)^{-t-1}R_t}{\displaystyle\sum_{t=1}^{n}(1+i)^{-t}R_t}
\end{aligned}$$

Also, it is important to note that duration for a 10-year mortgage is significantly shorter than for a 10-year bond. This happens because a mortgage repays some principal in every installment but the bond repays principal only at its redemption date. (This statement is not exactly true with premium and discount bonds, but the effect is generally not large enough to invalidate the above comparison between bonds and mortgages.)

4. Per dollar of dividend, formula (9.13) gives

$$\bar{d} = \frac{(Ia)_{\overline{\infty}|}}{a_{\overline{\infty}|}} = \frac{1.08/.08^2}{1/.08} = 13.5.$$

Note that this answer is independent of the dividend rate being paid on the preferred stock. Also note that duration for preferred stock is longer than for any of the other investments. This seems reasonable, since preferred stock involves payments being made into perpetuity.

9.9 IMMUNIZATION

Until this point in the book we have largely been analyzing individual transactions. We now shift our focus and consider a whole collection of transactions. More specifically, we consider the interrelationship between assets and liabilities for some financial enterprise, such as a bank, an insurance company, or a pension fund.

The assets will generate a series of cash inflows. We label the cash inflows as $A_1, A_2, \ldots, A_n$ which are made at times $1, 2, \ldots, n$. Similarly, the liabilities will generate a series of cash outflows. These are labelled $L_1, L_2, \ldots, L_n$ and occur at times $1, 2, \ldots, n$. The issue is how to achieve an equilibrium or safe balance between these cash inflows and outflows.

Before going further with the development of the theory, let us consider the nature of the problems that can arise if such equilibrium does not exist. The primary problem is the risk of adverse effects created by changes in the level of interest rates.

We can illustrate this problem with an example. Consider a financial institution, such as a bank or an insurance company, which issues a one-year instrument with a guaranteed rate of interest, e.g. a certificate of deposit (CD) or a guaranteed investment contract (GIC). A significant risk to the financial institution exists if the assets backing these contracts are invested either "too long" or "too short."

First, consider the case in which the assets are invested "too long," e.g. in investments with duration equal to two. The financial institution is vulnerable to losses if interest rates rise. Under these conditions contract holders are likely to withdraw their funds at the end of the year. The financial institution may

have to sell assets to pay these departing contract holders. However, the assets that would have to be sold have declined in value due to the rise in interest rates. Losses may be incurred as a result.

Second, consider the case in which the assets are invested "too short," e.g. in very short-term investments with duration close to zero. Now the financial institution is vulnerable to losses if interest rates fall. Since its assets are invested very short-term, its interest earnings will decline quickly and may not be sufficient to pay the guaranteed interest on the contracts at the end of the year. Thus, losses may be incurred in this case as well.

Immunization is a technique that has been developed to structure the assets and liabilities in a manner that would reduce or even eliminate the type of problem illustrated above. In other words, the enterprise would be "immunized" against the adverse effects created by changes in the level of interest rates.

Let the net receipt at time t be denoted by R_t, i.e.

$$R_t = A_t - L_t \text{ for } t = 1, 2, \ldots, n. \tag{9.18}$$

We will assume that the present value of the cash inflows from the assets is equal to the present value of the cash outflows from the liabilities. Thus, from formula (9.14) we have

$$P(i) = 0. \tag{9.19}$$

We now institute a small change in the rate of interest from i to $i + \epsilon$. If we expand $P(i)$ as a Taylor series as far as second derivatives, we have

$$P(i + \epsilon) = P(i) + \epsilon P'(i) + \frac{\epsilon^2}{2} P''(i + \xi), \text{ where } 0 < |\xi| < |\epsilon|.$$

Now $P(i)$ will have a local minimum at i if two conditions hold. First, we must have

$$P'(i) = 0. \tag{9.20}$$

Note that formula (9.20) can be interpreted as requiring that modified duration of the net receipts must equal zero. Second, we must have

$$P''(i) > 0. \tag{9.21}$$

If $P(i)$ has a local minimum, we obtain a key result in immunization theory; namely, that small changes in the interest rate in either direction will increase the present value of receipts. This certainly seems to be a highly desirable result, if it can be achieved.

The second derivative of $P(i)$ is used to define the property of *convexity*, denoted by $\bar{c}$, as

$$\bar{c} = \frac{P''(i)}{P(i)} \tag{9.22}$$

Formula (9.20) gives us the equation

$$P'(.1) = -\frac{2y}{1.1} + \frac{1000}{1.1} = 0.$$

This gives us two equations in two unknowns, so that we do not have the flexibility in this case to try to apply formula (9.21). Solving the two equations in two unknowns gives us $x = 500$ and $y = 500$.

Although we could not apply formula (9.21) in developing the investment allocation, we need to test our answer for convexity. We have

$$P''(.1) = \frac{(7.26)(500)}{(1.1)^4} - \frac{2200}{(1.1)^3} = 826.45 > 0.$$

Thus, our strategy satisfies formula (9.21) anyway, which is a happy result.

Let us test the result empirically for changes in i of 1% in either direction. We have

$$P(.10) = 0$$

$$P(.11) = 500 + \frac{(1.21)(500)}{(1.11)^2} - \frac{1100}{1.11} = .0406 > 0$$

$$P(.09) = 500 + \frac{(1.21)(500)}{(1.09)^2} - \frac{1100}{1.09} = .0421 > 0.$$

Thus, the value of $P(i)$ increases for movements of i away from $i = 10\%$ in either direction. This may seem too good to be true, but it illustrates a result that immunization is attempting to achieve.

Example 9.8 For the assets in Example 9.7 compute: (1) the modified duration, and (2) the convexity.

1. Using the asset portion of the expressions above, we have

$$P(i) = x + 1.21y(1 + i)^{-2} = 1000.00$$

and

$$P'(i) = -2.42y(1 + i)^{-3} = -909.09$$

when evaluated at $x = 500$, $y = 500$, and $i = .1$. Then applying formula (9.15) we have

$$\bar{v} = -\frac{P'(i)}{P(i)} = .90909.$$

Another approach is to use formula (9.16). We have the following modified durations:

Money market fund: $\bar{v} = 0$

Two-year zero coupon bonds: $\bar{v} = \frac{2}{1.1}$

The weighted average of these two values reflecting our investment allocation is

$$.5(0) + .5\left[\frac{2}{1.1}\right] = .90909.$$

As an exercise the reader will be asked to confirm that this value equals the modified duration of the liability.

2. Again, using the asset portion of the expressions above, we have

$$P''(i) = 7.26y(1 + i)^{-4} = 2479.34$$

when evaluated at $y = 500$ and $i = .1$. Then applying formula (9.22), we have

$$\bar{c} = \frac{P''(i)}{P(i)} = 2.47934.$$

As an exercise the reader will be asked to confirm that this value is greater than the convexity of the liability.

Example 9.9 For a 30-year home mortgage with level payments and an interest rate of 10.2% convertible monthly, find: (1) the modified duration of the payments, and (2) the convexity of the payments.

1. The monthly rate of interest is .102/12 = .0085. Then per dollar of monthly payment, we have

$$P(i) = \sum_{t=1}^{n} (1 + i)^{-t}$$

and

$$P'(i) = -\sum_{t=1}^{n} t(1 + i)^{-t-1}$$

so that

$$\bar{v} = -\frac{P'(i)}{P(i)} = v\frac{(Ia)_{\overline{n}|}}{a_{\overline{n}|}}.$$

When we evaluate this expression for $n = 360$ and $i = .0085$, we obtain

$$\bar{v} = \frac{11{,}283.80}{(1.0085)(112.0591)} = 99.85.$$

Thus, the modified duration of the payments is just under 100 months of the 360-month term of the mortgage.

2. To measure the convexity of the payments we need to take second derivative of $P(i)$

$$P''(i) = \sum_{t=1}^{n} t(t + 1)(1 + i)^{-t-2} = v^2 \sum_{t=1}^{n} (t^2 + t)v^t.$$

In order to evaluate this second derivative, we need to be able to evaluate $\sum_{t=1}^{n} t^2 v^t$. Since this summation has 360 terms in it, direct calculation is

This pattern reflects the normal yield curve with positive slope. Thus, we already see one new feature in this model. Classical immunization theory does not recognize the term structure of interest rates.

Let s_1 and s_2 be the amount of funds withdrawn by the holders of the CDs at the ends of years 1 and 2, respectively. The reader should be careful to note that s_1 and s_2 are amounts withdrawn, not rates of withdrawal. For each dollar deposited in the CDs we have the equation of value

$$1 = (1.08)^{-1}s_1 + (1.08)^{-2}s_2$$

which can be expressed as

$$s_2 = (1.08)^2 - 1.08s_1.$$

Explicit recognition of future withdrawal rates is a second new feature in this model.

Let p_1 and p_2 equal the proportion of funds the bank invests in one-year and two-year notes, respectively. Clearly, $p_1 + p_2 = 1$. Let f be the forward (reinvestment) rate on one-year notes in the second year. Explicit consideration of reinvestment rates is a third new feature in this model. Finally, let A_2 equal the accumulated value of the bank's funds arising from this transaction at the end of the second year. Then we have

$$\begin{aligned} A_2 &= \left[p_1(1.08) - s_1\right](1+f) + \left[p_2(1.085)^2 - s_2\right] \\ &= \left[p_1(1.08) - s_1\right](1+f) + \left[(1-p_1)(1.085)^2 - (1.08)^2 + (1.08)s_1\right] \\ &= \left[(1.08)(1+f) - (1.085)^2\right]p_1 + s_1(.08 - f) + (1.085)^2 - (1.08)^2. \end{aligned}$$

We need to analyze the impact of fluctuations in f on the value of A_2 in order to develop a recommendation concerning the selection of p_1 (and thus p_2 also).

First, we consider the case of a decrease in interest rates. Suppose that f is equal to 7%. This will create a low rate of withdrawal by the holders of the CDs at the end of the first year, since there are incentives to leave funds in the CD. Assume the amount withdrawn s_1 is only 10 cents per dollar originally invested. We then have

$$A_2 = -.021625p_1 + .011825.$$

We want A_2 to be greater than 0, which will happen if $p_1 < .5468$.

Second, we consider the case of an increase in interest rates. Suppose that f is equal to 9.5%. This will create a high rate of withdrawal by the holders of the CDs at the end of the first year, since high rates are available elsewhere. Assume the amount withdrawn s_1 is 90 cents per dollar originally invested. We then have

$$A_2 = .005375p_1 - .002675.$$

We want A_2 to be greater than 0, which will happen if $p_1 > .4977$.

Thus, the recommended investment strategy for the bank based on these assumptions is to choose p_1 such that $.4977 < p_1 < .5468$. Since $p_2 = 1 - p_1$, the investment allocation is determined.

Application of this technique requires investment managers to make certain key assumptions which are not required in classical immunization. One key assumption is how high and how low to set the forward (reinvestment) rate in working through the two cases. A second key assumption is the level of withdrawal rates that will develop in the two cases. Unfortunately, the technique is sensitive to these assumptions, which means that relatively modest changes in the assumptions can result in significantly different allocation strategies. Nevertheless, despite this sensitivity, the technique has proven to be useful in practical applications.

EXERCISES

9.4 Recognition of inflation

1. The nominal rate of interest is 8% and the rate of inflation is 5%. A single deposit is invested for 10 years. Let:
   A = value of the investment at the end of 10 years measured in "constant dollars," i.e. in dollars valued at time 0.
   B = value of the investment at the end of 10 years computed at the real rate of interest.
   Find the ratio A/B.

2. Rework Exercise 1 assuming equal level deposits at the beginning of each year during the 10-year period instead of a single deposit.

3. Money is invested for five years in a savings account earning 7% effective. If the rate of inflation is 10%, find the percentage of purchasing power lost during the period of investment.

4. A worker invests a sum of money for 20 years for retirement in a fund earning 8% effective. Interest income is subject to a 25% income tax rate. On Basis A the money accumulates tax-free and is subject to income tax at the end of the period of investment. On Basis B the interest income is subject to income tax each year as it is earned. Find the ratio of the after-tax accumulation on Basis A to the after-tax accumulation on Basis B.

5. An employee has just retired after working exactly 25 years for one employer. The employee was hired at $10,000 per year and received annual pay raises of 4%. The employee is eligible to receive a pension based on salary computed on one of three bases as specified below.

f) Would you expect the difference between the two answers to become larger for longer term bonds?

17. Consider the bonds in Exercise 16 from a different vantage point. Assume an investor has the choice of buying two 5% bonds or one 10% bond. Either choice will produce an identical flow of coupons.
    a) Find the difference in the amounts originally invested.
    b) Find the difference in the payouts when the bonds mature.
    c) Verify that *(b)* is the accumulated value of *(a)* at the spot rate of interest for the term of the bonds.

18. Find the value of $\ddot{s}_{\overline{5}|}$ using spot rates of interest in which the spot rates in Table 9.2 apply for the first year and the entire yield curve shifts downward by .25% for each year thereafter through the end of the five-year period.

19. Based on the yield curve given in Table 9.2, find the following expected forward rates:
    a) 1-year deferred 2-year forward rate.
    b) 2-year deferred 3-year forward rate.

20. A 6-year bond with 6% annual coupons has a yield rate of 12% effective. A 6-year bond with 10% annual coupons has a yield rate of 8% effective. Find the 6-year spot rate.

21. An investor has \$100,000 to invest for three years. The account balance may be reinvested at the end of either the first or second years. The yield curve given in Table 9.2 applies for the first year. The entire yield curve shifts upward by 2% for the second year and another 2% for the third year. Considering all possible patterns of investment, find the minimum and maximum accumulation at the end of the three-year investment period. Answer to the nearest dollar.

9.8 Duration

22. Show that $\lim_{i \to \infty} \bar{d}$ is equal to the point in time at which the first payment is made.

23. You are given the following series of payments:
    (i) \$100 at times $t = 1, 3, 5, \ldots, 19$
    (ii) \$200 at times $t = 2, 4, 6, \ldots, 20$
    You are asked to determine the time t^* such that the present value of the series of payments is equal to the present value of a single payment of \$3000 made at time t^*. Derive an exact expression for t^* assuming $i > 0$.

24. Find the duration of a common stock which pays dividends at the end of each year, if it is assumed that each dividend is 4% greater than the prior dividend and the effective rate of interest is 8%.

25. Find an expression for the modified duration of $\bar{a}_{\overline{n}|}$.

26. Show that the modified duration of a perpetuity-immediate and the present value of the same perpetuity are equal.

27. If there is only one payment, then the variance σ^2 in formula (9.17) would be zero. Verbally interpret the result that $\frac{d\bar{d}}{di} = 0$ in this situation.

28. A loan is to be repaid with payments of \$1000 at the end of one year, \$2000 at the end of two years, and \$3000 at the end of three years. The effective rate of interest is 25%.
    a) Find the amount of the loan.
    b) Find the duration.
    c) Find the modified duration.
    d) Find the variance σ^2 in formula (9.17).

9.9 Immunization

29. *a*) Confirm that the modified duration of the assets obtained in Example 9.8(1) is equal to the modified duration of the liability.
    b) Confirm that the convexity of the assets obtained in Example 9.8(2) is greater than the convexity of the liability.

30. Find the present value at 8% convertible semiannually of a 10-year annuity-immediate whose successive semiannual payments are $1, 4, 9, \ldots, 400$. Answer to the nearest dollar.

31. Derive the following relationship between modified duration and convexity:

$$\frac{d}{di}\bar{v} = \bar{v}^2 - \bar{c}.$$

32. The following relationship holds between convexity $\bar{c}$ and the variance σ^2 in formula (9.17):

$$\bar{c} = a(i)\sigma^2 + b(i).$$

    Find the functions $a(i)$ and $b(i)$.

33. *a*) In Example 9.7 assume that the investor puts \$600 in the money market fund and \$400 in two-year bonds. Find:
        (1) $P(.09)$
        (2) $P(.10)$
        (3) $P(.11)$
    b) Rework (*a*) assuming the investor puts \$400 in the money market fund and \$600 in two-year bonds.
    c) Verbally interpret the answers to (*a*) and (*b*) in comparison with the answers in Example 9.7.

34. Find the convexity of the following investments, assuming the effective rate of interest is 8%:
    a) A money market fund.

10

Stochastic approaches to interest

10.1 INTRODUCTION

The prior chapters in this book have presented the theory of interest on largely a deterministic basis. Chapter 10 introduces the reader to some stochastic approaches to interest.

Actually, there were two areas in the prior chapters where probabilities were explicitly introduced. One area was the consideration of default probabilities in Section 9.5, and the second was the assumption regarding withdrawals from the certificate of deposit illustration in Section 9.10. Also, formula (9.17) for the derivative of duration was interpreted in terms of a variance. Finally, there was implicit recognition of risk and uncertainty in such areas as the exercise of the call provision on callable bonds and the prepayment rates on mortgage loans. Nevertheless, the overall approach taken previously in the book has been largely deterministic.

In Chapter 10 we first consider the rate of interest directly as a random variable. We then provide a basic introduction to several models with a stochastic basis which have important applications in practice. Finally, we examine the widely used approach of scenario testing as a means of dealing with future uncertainty.

We assume that the reader has a basic background in mathematical statistics. The level of statistics used in this chapter is not advanced. Readers who would like a condensed refresher of the results from statistics that are used in this chapter are referred to Appendix IV.

10.2 INDEPENDENT RATES OF INTEREST

We now consider the rate of interest to be a random variable. In Section 10.2 we consider the case in which the rate of interest in one period is independent of the rate of interest in any other period. In Section 10.3 we will remove the independence assumption and examine some results when successive rates are correlated in some fashion.

A preliminary illustration

We will first demonstrate that expected accumulated and present values are not necessarily equal to accumulated and present values at the expected rate of interest. In order to demonstrate this possibility, consider the investment of one unit for ten years at an effective rate of interest which is unknown, but is equally likely to be 7%, 8%, or 9%. From elementary statistics the expected rate of interest is given by

$$E[i] = \frac{1}{3}[.07 + .08 + .09] = .08.$$

The expected accumulated value is given by

$$E[(1+i)^{10}] = \frac{1}{3}[(1.07)^{10} + (1.08)^{10} + (1.09)^{10}] = 2.16448.$$

However, the accumulated value at the expected rate of interest is

$$(1.08)^{10} = 2.15892.$$

Thus, in this illustration the expected accumulated value is not equal to the accumulated value at the expected rate of interest. The rate of interest which would produce the expected accumulated value can be determined from

$$(1+i)^{10} = 2.16448$$

which can be solved to give $i = .08028$, or 8.028%.

A similar result can also occur for present values. Assume we wish to invest an amount today sufficient to accumulate to one unit at the end of ten years and assume the same conditions described above concerning the unknown rate of interest.

The expected present value is given by

$$E[(1+i)^{-10}] = \frac{1}{3}[(1.07)^{-10} + (1.08)^{-10} + (1.09)^{-10}] = .46465$$

while the present value at the expected rate of interest is

$$(1.08)^{-10} = .46319.$$

The rate of interest which would produce the expected present value can be determined from

$$(1+i)^{-10} = .46465$$

$$\operatorname{var}[\,\ddot{s}_{\overline{n}|}\,] = \frac{m_2^s + m_1^s}{m_2^s - m_1^s}\,\ddot{s}_{\overline{n}|j} - \frac{2m_2^s}{m_2^s - m_1^s}\,\ddot{s}_{\overline{n}|i} - (\ddot{s}_{\overline{n}|i})^2 . \tag{10.8}$$

Present values

Parallel results to those above for accumulated values can also be developed for present values. However, we must be careful in the choice of interest rates to use, since in general

$$E\left[\frac{1}{1+i_t}\right] \neq \frac{1}{E[1+i_t]} .$$

Thus, when working with present values we must define i by

$$E[(1+i_t)^{-1}] = (1+i)^{-1} .$$

It must be stressed that this value of i is different than the value of i used above for accumulated values for which $E[i_t] = i$.

Our first result is to develop a formula for the mean of the present value of a single payment analogous to formula (10.1). We have

$$E\left[a^{-1}(n)\right] = (1+i)^{-n} . \tag{10.9}$$

The derivation of formula (10.9) is similar to formula (10.1) and is left as an exercise.

For the variance of the present value we have

$$\begin{aligned} \operatorname{var}[a^{-1}(n)] &= E[a^{-2}(n)] - \{E[a^{-1}(n)]\}^2 \\ &= (1+k)^{-n} - (1+i)^{-2n} \end{aligned} \tag{10.10}$$

where $(1+k)^{-1} = E[(1+i_t)^{-2}]$.

Unfortunately, this is as far as we can carry the development without knowing how i_t is distributed. The approach used above for finding the second moment for accumulated values will not work for present values. Evaluation of formula (10.10) will require computing the second moment based on a particular probability density function.

Now turning to the present value of an n-period annuity-immediate we have from Section 3.9

$$\begin{aligned} a_{\overline{n}|} &= (1+i_1)^{-1} + (1+i_1)^{-1}(1+i_2)^{-1} + \cdots \\ &\quad + (1+i_1)^{-1}(1+i_2)^{-1} \cdots (1+i_n)^{-1} = \sum_{t=1}^{n} \prod_{s=1}^{t} (1+i_s)^{-1} . \end{aligned} \tag{3.34}$$

The mean of $a_{\overline{n}|}$ is the anticipated result

$$E[\,a_{\overline{n}|}\,] = a_{\overline{n}|i} . \tag{10.11}$$

The derivation of formula (10.11) is similar to formula (10.5) and is left as an exercise.

Finally, we consider the variance of $a_{\overline{n}|}$. Define m_1^a and m_2^a to be the first and second moments of $(1 + i_t)^{-1}$ about the origin, respectively, i.e.

$$m_1^a = E[(1 + i_t)^{-1}] = (1 + i)^{-1} \tag{10.12}$$

and

$$m_2^a = E[(1 + i_t)^{-2}] = (1 + k)^{-1}. \tag{10.13}$$

Again, applying the results from Appendix IX, we have

$$\text{var}[a_{\overline{n}|}] = \frac{m_2^a + m_1^a}{m_2^a - m_1^a} a_{\overline{n}|k} - \frac{2m_2^a}{m_2^a - m_1^a} a_{\overline{n}|i} - (a_{\overline{n}|i})^2. \tag{10.14}$$

Finding numerical solutions

Thus, if we know the first and second moments of $1 + i_t$ and $(1 + i_t)^{-1}$, we can find the mean and variance of the accumulated value or the present value of either a single payment or a level annuity. We may choose to make an assumption about the probability density function for i_t. Unfortunately, the above formulas do not generally result in a known probability density function, even if we know the mean and the variance.

The standard approach to handle this situation in practice is to use *simulation*. The procedure is as follows:

1. Make an appropriate assumption about the probability density function for i_t and its parameters.
2. Generate a series of enough random numbers to run as many trials as desired. If m trials are desired, then mn random numbers will be necessary.
3. Using standard simulation techniques, use these random numbers to compute m sets of values for $i_1, i_2, \ldots, i_n$.
4. For each of the m sets $i_1, i_2, \ldots, i_n$ compute the required financial function, i.e. $a(n)$, $a^{-1}(n)$, $\ddot{s}_{\overline{n}|}$, $a_{\overline{n}|}$, or some other function.
5. The m outcomes can be used to develop an approximate probability density function for that financial function. Estimated probabilities for various possible outcomes can be calculated from these m outcomes.

We will not discuss the technique of simulation further, since we assume that the reader is familiar with simulation. Readers who are unfamiliar with the technique are referred to standard textbooks in mathematical statistics.

$$E[a(5)] = (1.067159)^5 = 1.38403.$$

Standard deviation

From formula (10.16) we have

$$\text{var}[1 + i_t] = e^{.12+.01}(e^{.01} - 1) = .011445.$$

Now $a(5)$ has a lognormal distribution with parameters 5μ and $5\sigma^2$, so that from formula (10.16)

$$\text{var}[a(5)] = e^{5(.12+.01)}(e^{.05} - 1) = .098212.$$

Thus, the standard deviation is $\sqrt{.098212} = .31339$. As an exercise, the reader will be asked to confirm this answer by the alternate approach of using formula (10.4*a*).

2. *Mean*

From formula (10.5) we have

$$E[\ddot{s}_{\overline{5}|}] = \ddot{s}_{\overline{5}|.067159} = 6.1023.$$

Standard deviation

From formula (10.8) we have

$$\text{var}[\ddot{s}_{\overline{5}|}] = .881737.$$

Thus, the standard deviation is $\sqrt{.881737} = .9390$.

3. *Mean*

We first must analyze the distribution of $(1 + i_t)^{-1}$. We have $\log_e(1 + i_t)^{-1} = -\log_e(1 + i_t)$, which has a normal distribution with mean $-\mu$ and variance σ^2. Thus, $(1 + i_t)^{-1}$ has a lognormal distribution with parameters $-\mu$ and σ^2.

From formula (10.15) we have

$$E[(1 + i_t)^{-1}] = e^{-.06+.005} = .946485.$$

Applying formula (10.9) gives us the mean

$$E[a^{-1}(5)] = (.946485)^5 = .759572.$$

Note that $(1 + i)^{-1} = .946485$, so that $i = .056541$, which is a different rate of interest than in 1 and 2 above.

Standard deviation

From formula (10.16) we have

$$\text{var}[(1 + i_t)^{-1}] = e^{-.12+.01}(e^{.01} - 1) = .009003.$$

Now $a^{-1}(5)$ has a lognormal distribution with parameters -5μ and $5\sigma^2$, so that from formula (10.16)

$$\text{var}[a^{-1}(5)] = e^{5(-.12+.01)}(e^{.05} - 1) = .029581.$$

Thus, the standard deviation is $\sqrt{.029581} = .17199$.

4. *Mean*

In 3 above we found that $i = .056541$. Thus, from formula (10.11) we have

$$E[a_{\overline{3}|}] = a_{\overline{3}|.056541} = 4.2523.$$

Standard deviation

We have $(1 + k)^{-1} = e^{-.12+.01}e^{.01} = e^{-.10} = .904837$, so that $k = .105171$. Applying formula (10.14) we have

$$\text{var}[a_{\overline{3}|}] = .383244.$$

Thus, the standard deviation is $\sqrt{.383244} = .6191$.

Example 10.4 Assume that $E[i_t] = .08$ for $t = 1, 2, 3$. Also assume that $1 + i_t$ follows a lognormal distribution with $\sigma^2 = .0001$. Find a 95% confidence interval for the accumulated value of an investment of 1 at the end of three years.

The random variable $\log_e(1 + i_t)$ has a normal distribution with parameters

$$\mu = \log_e(1.08) = .076961$$

and

$$\sigma^2 = .0001.$$

Applying formula (10.17), we find that $\log_e a(3)$ has a normal distribution with parameters

$$\mu = 3(.076961) = .230883$$

and

$$\sigma^2 = 3(.0001) = .0003.$$

The standard 95% confidence interval for $\log_e a(3)$ is given by

$$\mu \pm 1.96\sigma = .230883 \pm 1.96\sqrt{.0003}$$

or (.196935, .264831). The corresponding 95% confidence interval for $a(3)$ would be $(e^{.196935}, e^{.264831})$ or (1.21766, 1.30321). This is a 95% confidence interval around $(1.08)^3 = 1.25971$.

10.3 DEPENDENT RATES OF INTEREST

In Section 10.2 we assumed that the rates of interest i_t in each successive period are independent. In Section 10.3 we consider some results if this assumption is removed.

Dependent rates certainly have intuitive appeal. For example, if the rate of interest in one period is considerably higher than a long-term average rate, then it is reasonable to assume that the following period's rate is more likely to be higher than average than it is to be lower than average. The same assumption would also seem reasonable for rates lower than average.

$$k_1 + k_2 < 1$$
$$k_2 - k_1 < 1$$
$$-1 < k_2 < 1.$$

Again, note that if $k_1 = k_2 = 0$, we have independence and the results of Section 10.2 apply.

The derivations of the formulas for the variance and covariance of the AR(1) and AR(2) processes are given in the books by G.E.P. Box and G. M. Jenkins (1976) and R. B. Miller and D. W. Weichern (1977) listed in the bibliography. Formulas for the values of various financial functions based on the AR(1) and AR(2) processes are developed in two extensive papers by H. H. Panjer and D. R. Bellhouse (1980 and 1981) listed in the bibliography. These results are quite involved and are beyond the scope of this book.

The above cited approach to implementing the AR(1) and AR(2) processes are analytical in nature. However, another approach which can readily be used is simulation. Other models involving dependency can also be constructed and then implemented with simulation. It is important that any model be tested with empirical data before using it to see how well the model handles actual known results.

***Example 10.5** Assume that the long-term average effective rate of interest in a particular situation is 6% and that the prior year's rate is 9%. Compare the pattern produced by formula (10.19) over a three-year period, assuming each year's rate turns out to exactly equal its estimated rate based on the long-term average rate and the prior year's rate: (1) if $k = .2$, and (2) if $k = .8$.*

1. We have

$$i_1 = .06 + .2(.09 - .06) = .066$$
$$i_2 = .06 + .2(.066 - .06) = .0612$$
$$i_3 = .06 + .2(.0612 - .06) = .06024.$$

The regression of values to the mean rate of 6% is quite rapid.

2. We have

$$i_1 = .06 + .8(.09 - .06) = .084$$
$$i_2 = .06 + .8(.084 - .06) = .0792$$
$$i_3 = .06 + .8(.0792 - .06) = .07536.$$

Here the regression is much slower.

The purpose of this example is to illustrate the implicit pattern produced by different values of k. Of course, in practice each year's actual rate will quite likely differ from the estimated rate based on the long-term average rate and the prior year's rate, so that the smooth pattern above will be much more bumpy. However, the effect of the choice of k would be just as pronounced in terms of the magnitude of the regression toward the long-term average rate as in the above illustration.

***Example 10.6** It is known that $\delta_{[t]}$ follows an AR(1) process with mean = .09, variance = .003, and covariance between adjacent values = .002. The estimated value of $\delta_{[4]}$ is .075. Find the actual value of $\delta_{[3]}$.*

We divide formula (10.23) by formula (10.22) to obtain

$$\frac{\text{cov}}{\text{var}} = k = \frac{.002}{.003} = \frac{2}{3}.$$

We will denote actual values by δ^A and estimated values by δ^E. From formula (10.21) we have

$$\delta^E_{[4]} = \delta + k(\delta^A_{[3]} - \delta)$$
$$.075 = .09 + \frac{2}{3}(\delta^A_{[3]} - .09)$$

which gives

$$\delta^A_{[3]} = .0675.$$

***Example 10.7** It is known that $\delta_{[t]}$ follows an AR(2) process with a mean equal to .08. The following values are given:*

Z	*Actual* $\delta_{[z]}$	*Estimated* $\delta_{[z]}$
1	*.100*	*.086*
2	*.110*	*.094*
3	*.090*	*.102*
4	*.095*	*.092*

***Find the estimated value for** $\delta_{[5]}$.*

If we apply formula (10.24) for $z = 3$ and 4, we have

$$.102 = .08 + k_1(.11 - .08) + k_2(.10 - .08)$$

and

$$.092 = .08 + k_1(.09 - .08) + k_2(.11 - .08)$$

which simplifies to

$$.03k_1 + .02k_2 = .022$$
$$.01k_1 + .03k_2 = .012.$$

These are two equations in two unknowns, which can be solved to give $k_1 = .6$ and $k_2 = .2$. Note that the values of k satisfy the three necessary conditions. Thus, we have

$$\delta^E_{[5]} = .08 + .6(.095 - .08) + .2(.09 - .08) = .091.$$

10.4 AN ASSET PRICING MODEL

In Section 9.5 we discussed the effect which risk and uncertainty have on the rate of interest. In this section we extend that discussion and introduce the

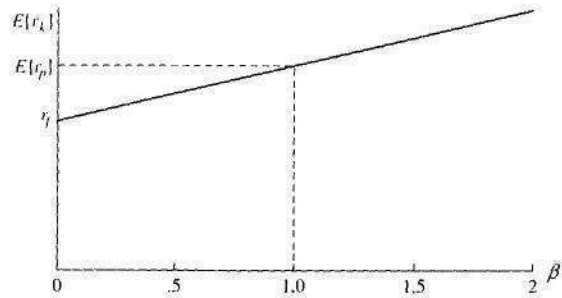

Figure 10.1 Graphical representation of the Capital Asset Pricing Model

When $\beta_k = 1$ the covariance in the numerator equals the variance in the denominator. This condition can be interpreted as the security k having the same level of risk as the market portfolio. The interpretations for $0 < \beta_k < 1$ and $\beta_k > 1$ reflect below-average or above-average risk, respectively.

As previously indicated, the Capital Asset Pricing Model was designed to estimate the expected yield rates on common stocks. Thus, the risk premium for the market portfolio, $E[r_p] - r_f$, is usually assumed to be a value close to 8.4% as indicated in Table 10.1.

The model has been subjected to extensive empirical testing and has performed reasonably well, particularly in view of its simplicity. However, it has been shown that other factors in addition to systematic risk have an influence on actual realized yield rates, e.g. seasonal factors and size factors. Also, the systematic risk for particular securities has been found to vary over time.

Nevertheless, despite these imperfections, the model has proven to be quite useful in practical applications. In fact, the model is so widely used that values of β_k are calculated for a large number of common stocks which are periodically updated and published. They are available from any major brokerage firm.

Although the Capital Asset Pricing Model is generally used in connection with common stock analysis, it has broader applicability. For example, we could use a risk premium for corporate bonds and apply it to individual corporate bonds. From Table 10.1 the appropriate risk premium for the market portfolio in this case would be approximately 1.7%. Some analysts might prefer to use a rate based on more recent experience than going back as far as 1926. Since bond prices are considerably less volatile than common stock

prices (see Table 10.2), a shorter period to use to determine the average experience would not seem to be unreasonable.

In Section 9.5 we discussed two alternative ways of reflecting risk: (1) through a risk premium in the interest rate, and (2) through an adjustment in the expected payments. It is possible to use the results above to develop an analytical approach to this second alternative.

To illustrate such an approach, consider an investment for one period. Let $E[W]$ be the expected value of the uncertain cash flow at the end of the period, which we discount at the yield rate r which includes an appropriate risk premium. Then the present value, denoted by V is

$$V = \frac{E[W]}{1 + r}. \tag{10.31}$$

We now seek to quantify the risk in the payment instead of in the rate of interest and then discount at the risk-free rate of interest. Let the equivalent adjusted payment, which is assumed certain to be paid, be denoted by W'. We then have

$$V = \frac{W'}{1 + r_f}. \tag{10.32}$$

We seek to find an expression for W' and establish that $W' < E[W]$.

In order to simplify the notation, we will set $E[r_k] = r$, $E[r_p] = \mu_p$, and $\text{var}[r_p] = \sigma_p^2$. We substitute formula (10.29) into formula (10.31) to obtain

$$\frac{E[W]}{V} = 1 + r = 1 + r_f + \beta_k(\mu_p - r_f).$$

Substituting formula (10.30) the expression becomes

$$1 + r_f + \frac{\text{cov}[r, r_p]}{\sigma_p^2}(\mu_p - r_f).$$

Now W is an uncertain cash flow, but V is fixed and does not covary with r_p. Thus, the expression becomes

$$1 + r_f + \text{cov}\left[\frac{W}{V} - 1, r_p\right] \frac{\mu_p - r_f}{\sigma_p^2}$$

and making a scalar adjustment we have

$$1 + r_f + \frac{\text{cov}[W, r_p]}{V} \cdot \frac{\mu_p - r_f}{\sigma_p^2}.$$

We define

Prior to expiry the call will sell at prices, which vary directly with the stock price, above the solid line. As expiry approaches, the call price will move closer to the solid line and will reach it at expiry for the value corresponding to the stock price at that time. For example, at expiry if the stock is selling for \$60, the call is worth \$10. If the stock is selling for \$40, the call is worthless.

Consider next a put which gives its owner the right to sell a share of the associated stock for \$50 at its expiry. Figure 10.3 gives a graphical illustration of the value of such a put at expiry.

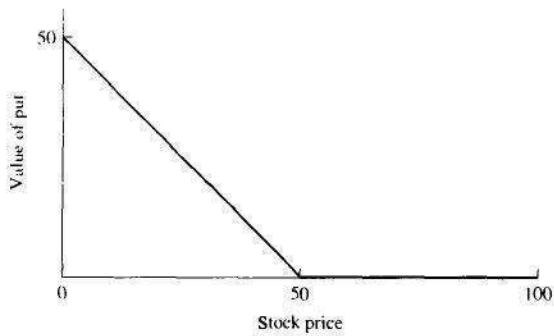

Figure 10.3 Value of an illustrative put at expiry

Prior to expiry the put will sell at prices, which vary inversely with the stock price, above the solid line. As expiry approaches, the put price will move closer to the solid line and will reach it at expiry for the value corresponding to the stock price at that time. For example, at expiry if the stock is selling for \$40, the put is worth \$10. If the stock is selling for \$60, the put is worthless.

As noted in Section 8.8, investors can either buy or sell options. Thus, if an investor thinks the security price is likely to rise, then a call should be purchased or a put should be sold. Conversely, if an investor thinks the security price is likely to fall, then a put should be purchased or a call should be sold.

It is possible to show that the value of a put and a call with the same exercise price on a given stock are related. Figure 10.4 is a graph of the results of buying a share of the stock underlying Figures 10.2 and 10.3.

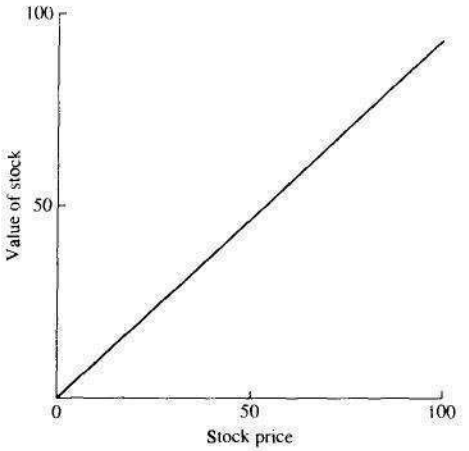

Figure 10.4 Value of the underlying stock

Now consider what happens if the investor purchases a share of stock and a put. The payoff at the expiry date is the sum of Figures 10.3 and 10.4 and is illustrated in Figure 10.5. However, Figure 10.5 is exactly the same as Figure 10.2, only it is \$50 higher at all points.

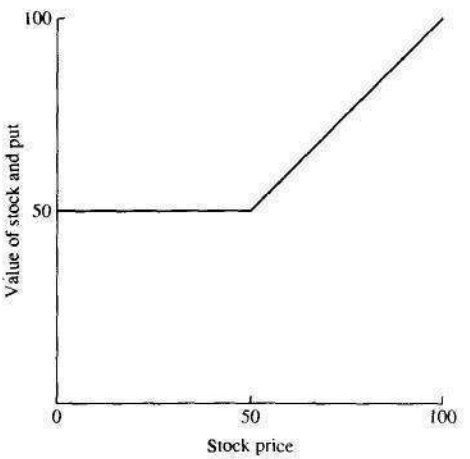

Figure 10.5 Value of the combination of a stock and a put at expiry

$$d_2 = \frac{\log_e(S/E) + (\delta - \frac{1}{2}\sigma^2)n}{\sigma\sqrt{n}} \qquad (10.39b)$$

σ = standard deviation of the force of interest on the stock or other underlying asset

Interestingly enough, the Black-Scholes formula is based on the assumption that $1 + i_t$ follows the lognormal distribution, just as described in Sections 10.2 and 10.3. Thus, σ as just defined is the standard deviation for $\delta_{[t]}$, as defined in formula (10.18).

Formula (10.38) also assumes that the underlying stock or other asset does not pay dividends prior to the expiry date. The derivation of formula (10.38) is lengthy and is given in Appendix X.

Since an American call can be exercised at the expiry date and at any time prior to that date, it would seem that its value should be greater than that for a European call. However, on a non-dividend paying stock, early exercise is never optimal. This can be seen from Figure 10.2, since the call prior to expiry sells at prices above the solid line. However, for a stock which pays dividends, early exercise may be optimal behavior in order to obtain the income from the dividend.

The corresponding formula for the value of a European put is given by

$$P = Ee^{-\delta n}[1 - N(d_2)] - S[1 - N(d_1)] \qquad (10.40)$$

where P is the value of the put and all other symbols are defined the same as in formula (10.38). Surprisingly, the value of an American put is greater than the value of a European put. The early exercise of a put can be optimal even when the underlying stock pays no dividends. On the one hand, a longer-term put may be worth more than a shorter-term put, since more time exists for a favorable movement in the underlying stock price. However, on the other hand, a shorter-term put may be worth more than a longer-term put due to the time value of money. The right to sell something at a favorable price in the future is worth less the more distant the future date. Thus, early exercise of a put may be optimal, so an American put is worth more than a European put.

Empirical tests of the Black-Scholes formula suggest that the formula works reasonably well under certain conditions. It tends to develop significant errors in the following situations:

1. When the exercise price E is far from the current market price S.
2. For securities with volatility much above or below average, i.e. σ being quite high or low.
3. If the expiry date is far in the future, i.e. for high values of n.

Example 10.9 Use the Black-Scholes formula to find the value of a European call expiring in one year with an exercise price of $100 for a stock currently selling for $90. The standard deviation for the continuous rate of return for this stock is .3, and the risk-free force of interest is 10%.

Applying formulas (10.39*a*) and (10.39*b*) we have

$$d_1 = \frac{\log_e(90/100) + (.1 + .09/2)(1)}{.3(1)} = .132$$

and

$$d_2 = \frac{\log_e(90/100) + (.1 - .09/2)(1)}{.3(1)} = -.168.$$

From tables for the cumulative normal distribution we have

$$N(.132) = .5525$$

and

$$N(-.168) = .4333.$$

Now substituting into formula (10.38)

$$C = (90)(.5525) - (100)(e^{-.1})(.4333) = \$10.52.$$

10.6 A RANDOM WALK MODEL

In Section 10.5 we discussed the Black-Scholes Option Pricing Model. In this section we introduce an alternative approach, the *binomial model*.

This model is based on the premise that security prices move in a *random walk*. In a random walk the probabilities of price movements of a given magnitude up or down from period to period remain unchanged and, moreover, the outcome in each interval is independent of the outcomes in prior intervals. There is considerable empirical evidence that, in fact, security prices follow a random walk.

We will introduce the binomial model for option pricing, which is based on a random walk, with a simple illustration. A stock is currently selling for $70. We assume that in one year the price must either be $55 or $95. We seek to find the value of a European call which expires at the end of one year with an exercise price of $75. We will assume that the one-year rate of interest is 10%.

If the stock falls to $55, the call will be worthless. If the stock rises to $95, the call will be worth $20 at expiry.

Now compare these outcomes with those that would result from buying one share of stock and borrowing $50. If the stock falls to $55, we can sell it and use the proceeds to pay off the loan for $50(1.1) = $55 leaving a zero balance.

n applying formula (10.41) we determine the value of p as above for the one-)eriod case. All the other values are known and the formula can be evaluated. The formula may be tedious to calculate by hand, but can easily be evaluated)y computer.

The random walk sequence as illustrated in Figure 10.6 is often called a *lattice*. Figure 10.6 is an example of a *closed lattice*, which will have $n + 1$ possible outcomes after n periods. These are represented by the $n + 1$ terms in the binomial expansion in formula (10.41).

A more complex model is the *open lattice*, which is illustrated in Figure 10.7. An open lattice will result when the upside and downside factors are not reciprocals of each other as above. An open lattice is more complex and will have 2^n endpoints rather than $n + 1$ endpoints. However, it can still readily be evaluated by computer.

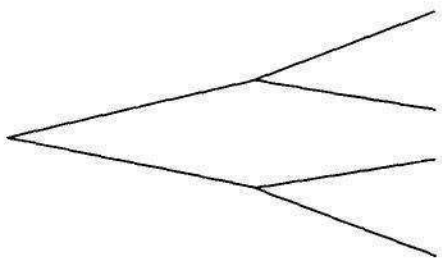

Figure 10.7 Illustrative open binomial lattice

In practice, the period until expiry is fixed. Thus, in applying the binomial method as we increase n we apply the method more times over shorter intervals. For example, if the period until expiry is one year, we could choose $n = 1, 2, 4, 12, 52, 365$ depending on whether we were considering prices changes annually, semiannually, quarterly, monthly, weekly, or daily.

Although formula (10.41) is general in nature, it does not give us any guidance concerning a reasonable choice of k. It can be shown that an excellent choice of k is given by

$$k = e^{\sigma\sqrt{h}} - 1 \tag{10.42}$$

where σ is defined the same as in the Black-Scholes formula and h is the length of the time interval between successive steps. Under formula (10.42) securities with higher variances will have higher values of k, and conversely, which certainly is reasonable.

The derivation of formula (10.42) is based on the lognormal distribution exactly as assumed in the derivation of the Black-Scholes formula. In fact, it can be shown that the limit as $h \to 0$ of the binomial method under these

conditions gives the same answer as the Black-Scholes formula. These results are stated without proof. Readers interested in pursuing the derivation of these results are referred to papers by J. C. Cox, S. Ross, and M. Rubinstein (1979) and R. P. Clancy (1985) listed in the bibliography.

Example 10.10 Rework Example 10.9 using the binomial method applied in four steps, i.e. on a quarterly basis.

From formula (10.42) we have

$$k = e^{\sigma\sqrt{h}} - 1 = e^{.3\sqrt{.25}} - 1 = .16183.$$

We now need to compute p on a basis which eliminates risk-free arbitrage. Our upside value is $90(1 + k) = 104.565$ and our downside value is $90(1 + k)^{-1} = 77.464$. An equation of value to determine p is

$$104.565p + 77.464(1 - p) = 90e^{.25(.1)} = 92.278$$

which can be solved to give $p = .5466$.

We have $n + 1 = 5$ possible outcomes:

$$\begin{aligned} 90(1 + k)^4 &= 163.988 \\ 90(1 + k)^2 &= 121.486 \\ 90 &= 90.000 \\ 90(1 + k)^{-2} &= 66.674 \\ 90(1 + k)^{-4} &= 49.394 \end{aligned}$$

We now apply formula (10.41) to obtain

$$C = e^{-.1}[(.5466)^4(163.988 - 100) + \begin{bmatrix} 4 \\ 1 \end{bmatrix}(.5466)^3(.4534)(121.486 - 100)] = \$10.93.$$

Thus, the binomial method applied on a quarterly basis produces a value for the call of \$10.93, in comparison to the value of \$10.52 given by the Black-Scholes formula. If the binomial method is applied more times over shorter periods, the answer will approach that given by the Black-Scholes formula. This result is illustrated in the exercises.

10.7 SCENARIO TESTING

In practical applications for many of the situations described in Chapters 9 and 10 an approach called *scenario testing* has become quite popular. This technique is, in essence, a type of sophisticated simulation which requires extensive computer capability to implement, since the volume of calculations is typically quite high. In this approach, each "scenario" refers to a different pattern of future interest rates, usually called an *interest rate path.*

For example, consider an analysis involving the matching of assets and liabilities for a financial institution. It is necessary to estimate future cash inflows from the assets and future cash outflows from the liabilities.

These cash inflows and outflows may involve inherent uncertainty as to amount, probability of payment, and timing. It is important to recognize in the

the opposite will be true if $p < 1/2$. The volatility of results is controlled by the magnitude of the possible upward and downward movements.

The third type of method is a *stochastic method.* Probably the most common stochastic method used is to assume that successive values of $1 + i_t$ follow a lognormal distribution. This is consistent with the definition given in formula (10.18) and with the assumptions underlying the Black-Scholes formula. One advantage with the stochastic method is the ability to readily adjust for the volatility of the interest rate. It can be shown that if common assumptions are used throughout, then this method becomes the limiting case of the binomial lattice method as the intervals in the binomial lattice shrink to zero. This happens in much the same fashion as the manner in which the Black-Scholes formula becomes the limiting case of the random walk model for the pricing of options.

One problem with both the probabilistic and stochastic methods is that interest rates may appear in some of the interest rate paths after a number of periods which seem unreasonably high or unreasonably low. One way of dealing with this problem is to lower the assumed volatility assumption in the movement of interest rates. Another way is to put bounds on how high or low the rates can go. However, neither method of dealing with the problem is ideal.

Time series analysis is useful in connection with the construction of scenarios. This forces the analyst to consider past data in the construction of scenarios. However, it is also important to remember the admonition that "the future will not duplicate the past." Thus, some of the scenarios may well involve conditions that have not previously occurred.

Finally, it is important not to ignore the importance of intermediate results by focusing too heavily on results at the end of the projection period. For example, for a particular scenario if we show a positive result at the end of the projection period, but are in a significant negative position midway through the period, we may not have a favorable scenario. At a minimum, a closer analysis of this particular scenario is in order.

The above is a brief descriptive discussion of scenario testing. It is difficult to be very specific in discussing the technique, since there are probably as many different ways of conducting scenario testing as there are analysts doing it (maybe even more!). Nevertheless, this section should be useful in providing the reader with a general appreciation of the various methods used and some of the considerations involved.

A discussion of scenario testing in greater depth appears in a paper by M. F. Jetton (1988) listed in the bibliography. This paper includes more analytical consideration of the three methods described above.

Example 10.11 ***$1000 is invested for five years at an unknown force of interest which changes annually. It is assumed that the force of interest follows an AR(1) process in which $\delta = .08$, $k = .7$, and $\sigma = .005$. Five random numbers are generated which produce the following values from the standard normal distribution: $+.82$, -1.61, $+.43$, $+1.02$, $-.79$. The force of interest for the year just ending was .10. Find the accumulated value of the investment at the end of five years for this interest rate path.***

Five successive values of $\delta_{[t]}$ are generated using formula (10.21) as follows:

$$\delta_{[1]} = .08 + .7(.1000 - .08) + .82(.005) = .0981$$
$$\delta_{[2]} = .08 + .7(.0981 - .08) - 1.61(.005) = .0846$$
$$\delta_{[3]} = .08 + .7(.0846 - .08) + .43(.005) = .0854$$
$$\delta_{[4]} = .08 + .7(.0854 - .08) + 1.02(.005) = .0889$$
$$\delta_{[5]} = .08 + .7(.0889 - .08) - .79(.005) = .0823$$

Thus, the accumulated value at the end of five years for this interest rate path is

$$1000e^{.0981}e^{.0846}e^{.0854}e^{.0889}e^{.0823} = 1000e^{.4393} = \$1552$$

to the nearest dollar.

In an actual problem of this type a large number of interest rate paths would be generated in like fashion. Statistical measures, such as the mean and standard deviation, could be computed for the distribution of accumulated values produced by the simulation.

Example 10.12 ***An insurance company issues a one-year guaranteed investment contract (GIC) which credits an annual effective rate of 8.5%. The company decides to invest the proceeds from the sale of the GIC in three-month instruments at the beginning of each quarter. The rate of interest earned for the first quarter is 8.4% convertible quarterly. It is assumed that future interest rates move according to the random walk model in which the probability of an upward movement in the rate of interest is .4, and the amount of upward and downward movements each quarter are .5% and .4% convertible quarterly, respectively. Find the probability that the company will lose money on the GIC.***

The binomial lattice in this case is an open lattice as illustrated in Figure 10.7 and not the more typical closed lattice. Thus, there are $2^3 = 8$ interest rate paths.

We will work in quarterly rates of interest. The rate of interest for the first quarter is .021. The amount of each upward movement in the rate of interest is .00125, while the amount of downward movement is .001.

The results for the right interest rates paths are summarized in Table 10.3.

10.3 Dependent rates of interest

7. Estimate $\delta_{[t]}$ for $t = 6, 7, 8$ given the data in Example 10.6, assuming future actual values are equal to estimated values.

8. *a)* Show that formula (10.25) simplifies to formula (10.22) if $k_2 = 0$.
   b) Show that formula (10.26) simplifies to formula (10.23) if $k_2 = 0$.

9. Estimate σ^2 for the AR(2) process in Example 10.7 based on the data for $Z = 1, 2, 3, 4$, assuming errors are distributed normally with mean = 0 and assuming that the population variance is equal to the sample variance.

10. Assume that the value of σ^2 for the AR(2) process in Example 10.7 is actually equal to .0002.
    a) Find $\text{var}[\delta_{[t]}]$.
    b) Find $\text{cov}[\delta_{[t]}, \delta_{[t+2]}]$.

11. It is known that $\delta_{[t]}$ is normally distributed and follows as AR(1) process. The following values are given:

Z	*Actual* $\delta_{[z]}$	*Estimated* $\delta_{[z]}$
1	.100	.104
2	.105	.096
3	.095	.100

    a) Find $\delta^E_{[4]}$.
    b) If $\text{var}[\delta_{[t]}] = .0001$, find $\text{cov}[\delta_{[3]}, \delta_{[6]}]$.

12. An investment fund earned 6% effective during the past year. For each of the next two years the yield on the investment fund i_t is equally likely to be:

$$.02 + k(i_{t-1} - .06)$$
$$.06 + k(i_{t-1} - .06)$$
$$.10 + k(i_{t-1} - .06).$$

    a) Show that

$$E[a(2)] = (1.06)^2 + \frac{1}{3}(.0032)k.$$

    b) Show that

$$\text{var}[a(2)] = \frac{1}{9}(.02158336 + .02157312k + .01079168k^2).$$

10.4 An asset pricing model

13. Using the data in Tables 10.1 and 10.2, compute the market price of risk for:
    a) Common stocks.
    b) Corporate bonds.
    c) Treasury bills.

14. For the common stock in Example 10.8, find the amount by which the expected value of the price of the stock at the end of one year was increased by the assumption it was a stock with above average market risk.

15. *a)* Use the dividend discount model developed in Section 7.10 to find the implied annual rate of dividend increase for the common stock in Example 10.8, if present values are computed at the rate of interest produced by the Capital Asset Pricing Model.
    b) Is this implied annual rate of dividend increase consistent with the answer to Example 10.8?
    c) Assuming the real risk-free rate of interest is 3%, find the excess of the annual rate of dividend increase over the annual rate of inflation.

16. A business firm decides to use the Capital Asset Pricing Model to evaluate two projects A and B. Project A has normal risk with $\beta = 1$, while Project B has high risk with $\beta = 2$. Each project is expected to return the same dollar amount at the end of one year and nothing thereafter. The risk free rate of interest is 5% and the market risk premium is 7%. If the two projects are combined into one project, find β for the combined project.

17. Stock A has $\beta = .5$ and investors expect it to return 7%. Stock B has $\beta = 1.5$ and investors expect it to return 15%.
    a) Find the risk free rate of interest.
    b) Find σ_A/σ_B, assuming equal correlation coefficients between the market portfolio and Stocks A and B.

18. An investment is projected to return \$110 at the end of one year and \$121 at the end of two years. The risk-free rate of interest is 5%, the market risk premium is 10%, and β for this investment is .5.
    a) Find the present value of the investment at the risk adjusted rate.
    b) Find the certainty-equivalent dollar returns at the end of each year.

10.5 An option pricing model

19. Verify algebraically that the Black-Scholes formula for a call given by formula (10.38) and the Black-Scholes formula for a put given by formula (10.40) satisfy put-call parity.

20. Find the value of the European put with the same expiry date and exercise price as the European call in Example 10.9.

21. Find the following limiting values for the value of a call option:
    a) $S = 0$.
    b) S is very large in relation to E.
    c) $E = 0$.
    d) E is very large in relation to S.
    e) $n = 0$.
    f) n is very large.

22. In order to illustrate the sensitivity of the Black-Scholes formula, rework Example 10.9 changing the following parameters one-by-one while holding all the other parameters constant. (Note: You will need a table for values of the cumulative

Appendix I

Table of compound interest functions

Values of v^n, $(1+i)^n$, $a_{\overline{n}|}$, $s_{\overline{n}|}$, $1/s_{\overline{n}|}$ and a table of constants are tabulated at the following rates of interest: 0.5%, 1%, 1.5%, 2%, 2.5%, 3%, 3.5%, 4%, 4.5%, 5%, 6%, 7%, 8%, 9%, 10%, 12%.

INTEREST TABLES AT 0.5%

Constants	
Function	Value
i	.005000
$i^{(2)}$	.004994
$i^{(4)}$	.004991
$i^{(12)}$	.004989
δ	.004988
d	.004975
$d^{(2)}$	.004981
$d^{(4)}$	.004984
$d^{(12)}$	.004987
δ	.004988
v	.995025
$v^{1/2}$	.997509
$v^{1/4}$	.998754
$v^{1/12}$	.999584
$1+i$	1.005000
$(1+i)^{1/2}$	1.002497
$(1+i)^{1/4}$	1.001248
$(1+i)^{1/12}$	1.000416
$i/i^{(2)}$	1.001248
$i/i^{(4)}$	1.001873
$i/i^{(12)}$	1.002290
i/δ	1.002498
$i/d^{(2)}$	1.003748
$i/d^{(4)}$	1.003123
$i/d^{(12)}$	1.002706
i/δ	1.002498

n	v^n	$(1+i)^n$	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$	$1/s_{\overline{n}\rceil}$
1	.99502	1.00500	.9950	1.0000	1.000000
2	.99007	1.01003	1.9851	2.0050	.498753
3	.98515	1.01508	2.9702	3.0150	.331672
4	.98025	1.02015	3.9505	4.0301	.248133
5	.97537	1.02525	4.9259	5.0503	.198010
6	.97052	1.03038	5.8964	6.0755	.164595
7	.96569	1.03553	6.8621	7.1059	.140729
8	.96089	1.04071	7.8230	8.1414	.122829
9	.95610	1.04591	8.7791	9.1821	.108907
10	.95135	1.05114	9.7304	10.2280	.097771
11	.94661	1.05640	10.6770	11.2792	.088659
12	.94191	1.06168	11.6189	12.3356	.081066
13	.93722	1.06699	12.5562	13.3972	.074642
14	.93256	1.07232	13.4887	14.4642	.069136
15	.92792	1.07768	14.4166	15.5365	.064364
16	.92330	1.08307	15.3399	16.6142	.060189
17	.91871	1.08849	16.2586	17.6973	.056506
18	.91414	1.09393	17.1728	18.7858	.053232
19	.90959	1.09940	18.0824	19.8797	.050303
20	.90506	1.10490	18.9874	20.9791	.047666
21	.90056	1.11042	19.8880	22.0840	.045282
22	.89608	1.11597	20.7841	23.1944	.043114
23	.89162	1.12155	21.6757	24.3104	.041135
24	.88719	1.12716	22.5629	25.4320	.039321
25	.88277	1.13280	23.4456	26.5591	.037652
26	.87838	1.13846	24.3240	27.6919	.036112
27	.87401	1.14415	25.1980	28.8304	.034686
28	.86966	1.14987	26.0677	29.9745	.033362
29	.86533	1.15562	26.9330	31.1244	.032129
30	.86103	1.16140	27.7941	32.2800	.030979
31	.85675	1.16721	28.6508	33.4414	.029903
32	.85248	1.17304	29.5033	34.6086	.028895
33	.84824	1.17891	30.3515	35.7817	.027947
34	.84402	1.18480	31.1955	36.9606	.027056
35	.83982	1.19073	32.0354	38.1454	.026215
36	.83564	1.19668	32.8710	39.3361	.025422
37	.83149	1.20266	33.7025	40.5328	.024671
38	.82735	1.20868	34.5299	41.7354	.023960
39	.82323	1.21472	35.3531	42.9441	.023286
40	.81914	1.22079	36.1722	44.1588	.022646
41	.81506	1.22690	36.9873	45.3796	.022036
42	.81101	1.23303	37.7983	46.6065	.021456
43	.80697	1.23920	38.6053	47.8396	.020903
44	.80296	1.24539	39.4082	49.0788	.020375
45	.79896	1.25162	40.2072	50.3242	.019871
46	.79499	1.25788	41.0022	51.5758	.019389
47	.79103	1.26417	41.7932	52.8337	.018927
48	.78710	1.27049	42.5803	54.0978	.018485
49	.78318	1.27684	43.3635	55.3683	.018061
50	.77929	1.28323	44.1428	56.6452	.017654

INTEREST TABLES AT 2%

Constants	
Function	Value
i	.020000
$i^{(2)}$	.019901
$i^{(4)}$	.019852
$i^{(12)}$	.019819
δ	.019803
d	.019608
$d^{(2)}$	.019705
$d^{(4)}$	.019754
$d^{(12)}$	.019786
δ	.019803
v	.980392
$v^{1/2}$	.990148
$v^{1/4}$	.995062
$v^{1/12}$	.998351
$1+i$	1.020000
$(1+i)^{1/2}$	1.009950
$(1+i)^{1/4}$	1.004963
$(1+i)^{1/12}$	1.001652
$i/i^{(2)}$	1.004975
$i/i^{(4)}$	1.007469
$i/i^{(12)}$	1.009134
i/δ	1.009967
$i/d^{(2)}$	1.014975
$i/d^{(4)}$	1.012469
$i/d^{(12)}$	1.010801
i/δ	1.009967

n	v^n	$(1+i)^n$	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$	$1/s_{\overline{n}\rceil}$
1	.98039	1.02000	.9804	1.0000	1.000000
2	.96117	1.04040	1.9416	2.0200	.495050
3	.94232	1.06121	2.8839	3.0604	.326755
4	.92385	1.08243	3.8077	4.1216	.242624
5	.90573	1.10408	4.7135	5.2040	.192158
6	.88797	1.12616	5.6014	6.3081	.158526
7	.87056	1.14869	6.4720	7.4343	.134512
8	.85349	1.17166	7.3255	8.5830	.116510
9	.83676	1.19509	8.1622	9.7546	.102515
10	.82035	1.21899	8.9826	10.9497	.091327
11	.80426	1.24337	9.7868	12.1687	.082178
12	.78849	1.26824	10.5753	13.4121	.074560
13	.77303	1.29361	11.3484	14.6803	.068118
14	.75788	1.31948	12.1062	15.9739	.062602
15	.74301	1.34587	12.8493	17.2934	.057825
16	.72845	1.37279	13.5777	18.6393	.053650
17	.71416	1.40024	14.2919	20.0121	.049970
18	.70016	1.42825	14.9920	21.4123	.046702
19	.68643	1.45681	15.6785	22.8406	.043782
20	.67297	1.48595	16.3514	24.2974	.041157
21	.65978	1.51567	17.0112	25.7833	.038785
22	.64684	1.54598	17.6580	27.2990	.036631
23	.63416	1.57690	18.2922	28.8450	.034668
24	.62172	1.60844	18.9139	30.4219	.032871
25	.60953	1.64061	19.5235	32.0303	.031220
26	.59758	1.67342	20.1210	33.6709	.029699
27	.58586	1.70689	20.7069	35.3443	.028293
28	.57437	1.74102	21.2813	37.0512	.026990
29	.56311	1.77584	21.8444	38.7922	.025778
30	.55207	1.81136	22.3965	40.5681	.024650
31	.54125	1.84759	22.9377	42.3794	.023596
32	.53063	1.88454	23.4683	44.2270	.022611
33	.52023	1.92223	23.9886	46.1116	.021687
34	.51003	1.96068	24.4986	48.0338	.020819
35	.50003	1.99989	24.9986	49.9945	.020002
36	.49022	2.03989	25.4888	51.9944	.019233
37	.48061	2.08069	25.9695	54.0343	.018507
38	.47119	2.12230	26.4406	56.1149	.017821
39	.46195	2.16474	26.9026	58.2372	.017171
40	.45289	2.20804	27.3555	60.4020	.016556
41	.44401	2.25220	27.7995	62.6100	.015972
42	.43530	2.29724	28.2348	64.8622	.015417
43	.42677	2.34319	28.6616	67.1595	.014890
44	.41840	2.39005	29.0800	69.5027	.014388
45	.41020	2.43785	29.4902	71.8927	.013910
46	.40215	2.48661	29.8923	74.3306	.013453
47	.39427	2.53634	30.2866	76.8172	.013018
48	.38654	2.58707	30.6731	79.3535	.012602
49	.37896	2.63881	31.0521	81.9406	.012204
50	.37153	2.69159	31.4236	84.5794	.011823

Constants	
Function	Value
i	.025000
$i^{(2)}$	.024846
$i^{(4)}$	.024769
$i^{(12)}$	.024718
δ	.024693
d	.024390
$d^{(2)}$	.024541
$d^{(4)}$	.024617
$d^{(12)}$	.024667
δ	.024693
v	.975610
$v^{1/2}$	.987730
$v^{1/4}$	.993846
$v^{1/12}$	.997944
$1+i$	1.025000
$(1+i)^{1/2}$	1.012423
$(1+i)^{1/4}$	1.006192
$(1+i)^{1/12}$	1.002060
$i/i^{(2)}$	1.006211
$i/i^{(4)}$	1.009327
$i/i^{(12)}$	1.011407
i/δ	1.012449
$i/d^{(2)}$	1.018711
$i/d^{(4)}$	1.015577
$i/d^{(12)}$	1.013491
i/δ	1.012449

INTEREST TABLES AT 2.5%

n	v^n	$(1+i)^n$	$a_{\overline{n}\rceil}$	$s_{\overline{n}\rceil}$	$1/s_{\overline{n}\rceil}$
1	.97561	1.02500	.9756	1.0000	1.000000
2	.95181	1.05063	1.9274	2.0250	.493827
3	.92860	1.07689	2.8560	3.0756	.325137
4	.90595	1.10381	3.7620	4.1525	.240818
5	.88385	1.13141	4.6458	5.2563	.190247
6	.86230	1.15969	5.5081	6.3877	.156550
7	.84127	1.18869	6.3494	7.5474	.132495
8	.82075	1.21840	7.1701	8.7361	.114467
9	.80073	1.24886	7.9709	9.9545	.100457
10	.78120	1.28008	8.7521	11.2034	.089259
11	.76214	1.31209	9.5142	12.4835	.080106
12	.74356	1.34489	10.2578	13.7956	.072487
13	.72542	1.37851	10.9832	15.1404	.066048
14	.70773	1.41297	11.6909	16.5190	.060537
15	.69047	1.44830	12.3814	17.9319	.055766
16	.67362	1.48451	13.0550	19.3802	.051599
17	.65720	1.52162	13.7122	20.8647	.047928
18	.64117	1.55966	14.3534	22.3863	.044670
19	.62553	1.59865	14.9789	23.9460	.041761
20	.61027	1.63862	15.5892	25.5447	.039147
21	.59539	1.67958	16.1845	27.1833	.036787
22	.58086	1.72157	16.7654	28.8629	.034647
23	.56670	1.76461	17.3321	30.5844	.032696
24	.55288	1.80873	17.8850	32.3490	.030913
25	.53939	1.85394	18.4244	34.1578	.029276
26	.52623	1.90029	18.9506	36.0117	.027769
27	.51340	1.94780	19.4640	37.9120	.026377
28	.50088	1.99650	19.9649	39.8598	.025088
29	.48866	2.04641	20.4535	41.8563	.023891
30	.47674	2.09757	20.9303	43.9027	.022778
31	.46511	2.15001	21.3954	46.0003	.021739
32	.45377	2.20376	21.8492	48.1503	.020768
33	.44270	2.25885	22.2919	50.3540	.019859
34	.43191	2.31532	22.7238	52.6129	.019007
35	.42137	2.37321	23.1452	54.9282	.018206
36	.41109	2.43254	23.5563	57.3014	.017452
37	.40107	2.49335	23.9573	59.7339	.016741
38	.39128	2.55568	24.3486	62.2273	.016070
39	.38174	2.61957	24.7303	64.7830	.015436
40	.37243	2.68506	25.1028	67.4026	.014836
41	.36335	2.75219	25.4661	70.0876	.014268
42	.35448	2.82100	25.8206	72.8398	.013729
43	.34584	2.89152	26.1664	75.6608	.013217
44	.33740	2.96381	26.5038	78.5523	.012730
45	.32917	3.03790	26.8330	81.5161	.012268
46	.32115	3.11385	27.1542	84.5540	.011827
47	.31331	3.19170	27.4675	87.6679	.011407
48	.30567	3.27149	27.7732	90.8596	.011006
49	.29822	3.35328	28.0714	94.1311	.010623
50	.29094	3.43711	28.3623	97.4843	.010258

INTEREST TABLES AT 4%

Constants	
Function	Value
i	.040000
$i^{(2)}$	.039608
$i^{(4)}$	.039414
$i^{(12)}$	.039285
δ	.039221
d	.038462
$d^{(2)}$	.038839
$d^{(4)}$	.039029
$d^{(12)}$	.039157
δ	.039221
v	.961538
$v^{1/2}$	.980581
$v^{1/4}$	.990243
$v^{1/12}$	.996737
$1+i$	1.040000
$(1+i)^{1/2}$	1.019804
$(1+i)^{1/4}$	1.009853
$(1+i)^{1/12}$	1.003274
$i/i^{(2)}$	1.009902
$i/i^{(4)}$	1.014877
$i/i^{(12)}$	1.018204
i/δ	1.019869
$i/d^{(2)}$	1.029902
$i/d^{(4)}$	1.024877
$i/d^{(12)}$	1.021537
i/δ	1.019869

n	v^n	$(1+i)^n$	$a_{\overline{n}\|}$	$s_{\overline{n}\|}$	$1/s_{\overline{n}\|}$
1	.96154	1.04000	.9615	1.0000	1.000000
2	.92456	1.08160	1.8861	2.0400	.490196
3	.88900	1.12486	2.7751	3.1216	.320349
4	.85480	1.16986	3.6299	4.2465	.235490
5	.82193	1.21665	4.4518	5.4163	.184627
6	.79031	1.26532	5.2421	6.6330	.150762
7	.75992	1.31593	6.0021	7.8983	.126610
8	.73069	1.36857	6.7327	9.2142	.108528
9	.70259	1.42331	7.4353	10.5828	.094493
10	.67556	1.48024	8.1109	12.0061	.083291
11	.64958	1.53945	8.7605	13.4864	.074149
12	.62460	1.60103	9.3851	15.0258	.066552
13	.60057	1.66507	9.9856	16.6268	.060144
14	.57748	1.73168	10.5631	18.2919	.054669
15	.55526	1.80094	11.1184	20.0236	.049941
16	.53391	1.87298	11.6523	21.8245	.045820
17	.51337	1.94790	12.1657	23.6975	.042199
18	.49363	2.02582	12.6593	25.6454	.038993
19	.47464	2.10685	13.1339	27.6712	.036139
20	.45639	2.19112	13.5903	29.7781	.033582
21	.43883	2.27877	14.0292	31.9692	.031280
22	.42196	2.36992	14.4511	34.2480	.029199
23	.40573	2.46472	14.8568	36.6179	.027309
24	.39012	2.56330	15.2470	39.0826	.025587
25	.37512	2.66584	15.6221	41.6459	.024012
26	.36069	2.77247	15.9828	44.3117	.022567
27	.34682	2.88337	16.3296	47.0842	.021239
28	.33348	2.99870	16.6631	49.9676	.020013
29	.32065	3.11865	16.9837	52.9663	.018880
30	.30832	3.24340	17.2920	56.0849	.017830
31	.29646	3.37313	17.5885	59.3283	.016855
32	.28506	3.50806	17.8736	62.7015	.015949
33	.27409	3.64838	18.1476	66.2095	.015104
34	.26355	3.79432	18.4112	69.8579	.014315
35	.25342	3.94609	18.6646	73.6522	.013577
36	.24367	4.10393	18.9083	77.5983	.012887
37	.23430	4.26809	19.1426	81.7022	.012240
38	.22529	4.43881	19.3679	85.9703	.011632
39	.21662	4.61637	19.5845	90.4091	.011061
40	.20829	4.80102	19.7928	95.0255	.010523
41	.20028	4.99306	19.9931	99.8265	.010017
42	.19257	5.19278	20.1856	104.8196	.009540
43	.18517	5.40050	20.3708	110.0124	.009090
44	.17805	5.61652	20.5488	115.4129	.008665
45	.17120	5.84118	20.7200	121.0294	.008262
46	.16461	6.07482	20.8847	126.8706	.007882
47	.15828	6.31782	21.0429	132.9454	.007522
48	.15219	6.57053	21.1951	139.2632	.007181
49	.14634	6.83335	21.3415	145.8337	.006857
50	.14071	7.10668	21.4822	152.6671	.006550

Constants	
Function	Value
i	.045000
$i^{(2)}$	.044505
$i^{(4)}$	.044260
$i^{(12)}$	.044098
δ	.044017
d	.043062
$d^{(2)}$	.043536
$d^{(4)}$	.043776
$d^{(12)}$	.043936
δ	.044017
v	.956938
$v^{1/2}$	.978232
$v^{1/4}$	.989056
$v^{1/12}$	.996339
$1+i$	1.045000
$(1+i)^{1/2}$	1.022252
$(1+i)^{1/4}$	1.011065
$(1+i)^{1/12}$	1.003675
$i/i^{(2)}$	1.011126
$i/i^{(4)}$	1.016720
$i/i^{(12)}$	1.020461
i/δ	1.022335
$i/d^{(2)}$	1.033626
$i/d^{(4)}$	1.027970
$i/d^{(12)}$	1.024211
i/δ	1.022335

INTEREST TABLES AT 4.5%

n	v^n	$(1+i)^n$	$a_{\overline{n}\|}$	$s_{\overline{n}\|}$	$1/s_{\overline{n}\|}$
1	.95694	1.04500	.9569	1.0000	1.000000
2	.91573	1.09203	1.8727	2.0450	.488998
3	.87630	1.14117	2.7490	3.1370	.318773
4	.83856	1.19252	3.5875	4.2782	.233744
5	.80245	1.24618	4.3900	5.4707	.182792
6	.76790	1.30226	5.1579	6.7169	.148878
7	.73483	1.36086	5.8927	8.0192	.124701
8	.70319	1.42210	6.5959	9.3800	.106610
9	.67290	1.48610	7.2688	10.8021	.092574
10	.64393	1.55297	7.9127	12.2882	.081379
11	.61620	1.62285	8.5289	13.8412	.072248
12	.58966	1.69588	9.1186	15.4640	.064666
13	.56427	1.77220	9.6829	17.1599	.058275
14	.53997	1.85194	10.2228	18.9321	.052820
15	.51672	1.93528	10.7395	20.7841	.048114
16	.49447	2.02237	11.2340	22.7193	.044015
17	.47318	2.11338	11.7072	24.7417	.040418
18	.45280	2.20848	12.1600	26.8551	.037237
19	.43330	2.30786	12.5933	29.0636	.034407
20	.41464	2.41171	13.0079	31.3714	.031876
21	.39679	2.52024	13.4047	33.7831	.029601
22	.37970	2.63365	13.7844	36.3034	.027546
23	.36335	2.75217	14.1478	38.9370	.025682
24	.34770	2.87601	14.4955	41.6892	.023987
25	.33273	3.00543	14.8282	44.5652	.022439
26	.31840	3.14068	15.1466	47.5706	.021021
27	.30469	3.28201	15.4513	50.7113	.019719
28	.29157	3.42970	15.7429	53.9933	.018521
29	.27902	3.58404	16.0219	57.4230	.017415
30	.26700	3.74532	16.2889	61.0071	.016392
31	.25550	3.91386	16.5444	64.7524	.015443
32	.24450	4.08998	16.7889	68.6662	.014563
33	.23397	4.27403	17.0229	72.7562	.013745
34	.22390	4.46636	17.2468	77.0303	.012982
35	.21425	4.66735	17.4610	81.4966	.012270
36	.20503	4.87738	17.6660	86.1640	.011606
37	.19620	5.09686	17.8622	91.0413	.010984
38	.18775	5.32622	18.0500	96.1382	.010402
39	.17967	5.56590	18.2297	101.4644	.009856
40	.17193	5.81636	18.4016	107.0303	.009343
41	.16453	6.07810	18.5661	112.8467	.008862
42	.15744	6.35162	18.7235	118.9248	.008409
43	.15066	6.63744	18.8742	125.2764	.007982
44	.14417	6.93612	19.0184	131.9138	.007581
45	.13796	7.24825	19.1563	138.8500	.007202
46	.13202	7.57442	19.2884	146.0982	.006845
47	.12634	7.91527	19.4147	153.6726	.006507
48	.12090	8.27146	19.5356	161.5879	.006189
49	.11569	8.64367	19.6513	169.8594	.005887
50	.11071	9.03264	19.7620	178.5030	.005602

INTEREST TABLES AT 7%

Constants	
Function	Value
i	.070000
$i^{(2)}$	.068816
$i^{(4)}$	.068234
$i^{(12)}$	.067850
δ	.067659
d	.065421
$d^{(2)}$	.066527
$d^{(4)}$	.067090
$d^{(12)}$	.067468
δ	.067659
v	.934579
$v^{1/2}$	.966736
$v^{1/4}$	.983228
$v^{1/12}$	.994378
$1+i$	1.070000
$(1+i)^{1/2}$	1.034408
$(1+i)^{1/4}$	1.017059
$(1+i)^{1/12}$	1.005654
$i/i^{(2)}$	1.017204
$i/i^{(4)}$	1.025880
$i/i^{(12)}$	1.031691
i/δ	1.034605
$i/d^{(2)}$	1.052204
$i/d^{(4)}$	1.043380
$i/d^{(12)}$	1.037525
i/δ	1.034605

n	v^n	$(1+i)^n$	$a_{\overline{n}\|}$	$s_{\overline{n}\|}$	$1/s_{\overline{n}\|}$
1	.93458	1.07000	.9346	1.0000	1.000000
2	.87344	1.14490	1.8080	2.0700	.483092
3	.81630	1.22504	2.6243	3.2149	.311052
4	.76290	1.31080	3.3872	4.4399	.225228
5	.71299	1.40255	4.1002	5.7507	.173891
6	.66634	1.50073	4.7665	7.1533	.139796
7	.62275	1.60578	5.3893	8.6540	.115553
8	.58201	1.71819	5.9713	10.2598	.097468
9	.54393	1.83846	6.5152	11.9780	.083486
10	.50835	1.96715	7.0236	13.8164	.072378
11	.47509	2.10485	7.4987	15.7836	.063357
12	.44401	2.25219	7.9427	17.8885	.055902
13	.41496	2.40985	8.3577	20.1406	.049651
14	.38782	2.57853	8.7455	22.5505	.044345
15	.36245	2.75903	9.1079	25.1290	.039795
16	.33873	2.95216	9.4466	27.8881	.035858
17	.31657	3.15882	9.7632	30.8402	.032425
18	.29586	3.37993	10.0591	33.9990	.029413
19	.27651	3.61653	10.3356	37.3790	.026753
20	.25842	3.86968	10.5940	40.9955	.024393
21	.24151	4.14056	10.8355	44.8652	.022289
22	.22571	4.43040	11.0612	49.0057	.020406
23	.21095	4.74053	11.2722	53.4361	.018714
24	.19715	5.07237	11.4693	58.1767	.017189
25	.18425	5.42743	11.6536	63.2490	.015811
26	.17220	5.80735	11.8258	68.6765	.014561
27	.16093	6.21387	11.9867	74.4838	.013426
28	.15040	6.64884	12.1371	80.6977	.012392
29	.14056	7.11426	12.2777	87.3465	.011449
30	.13137	7.61226	12.4090	94.4608	.010586
31	.12277	8.14511	12.5318	102.0730	.009797
32	.11474	8.71527	12.6466	110.2182	.009073
33	.10723	9.32534	12.7538	118.9334	.008408
34	.10022	9.97811	12.8540	128.2588	.007797
35	.09366	10.67658	12.9477	138.2369	.007234
36	.08754	11.42394	13.0352	148.9135	.006715
37	.08181	12.22362	13.1170	160.3374	.006237
38	.07646	13.07927	13.1935	172.5610	.005795
39	.07146	13.99482	13.2649	185.6403	.005387
40	.06678	14.97446	13.3317	199.6351	.005009
41	.06241	16.02267	13.3941	214.6096	.004660
42	.05833	17.14426	13.4524	230.6322	.004336
43	.05451	18.34435	13.5070	247.7765	.004036
44	.05095	19.62846	13.5579	266.1209	.003758
45	.04761	21.00245	13.6055	285.7493	.003500
46	.04450	22.47262	13.6500	306.7518	.003260
47	.04159	24.04571	13.6916	329.2244	.003037
48	.03887	25.72891	13.7305	353.2701	.002831
49	.03632	27.52993	13.7668	378.9990	.002639
50	.03395	29.45703	13.8007	406.5289	.002460

INTEREST TABLES AT 8%

Constants	
Function	Value
i	.080000
$i^{(2)}$	.078461
$i^{(4)}$	.077706
$i^{(12)}$	.077208
δ	.076961
d	.074074
$d^{(2)}$	.075499
$d^{(4)}$	.076225
$d^{(12)}$	.076715
δ	.076961
v	.925926
$v^{1/2}$	.962250
$v^{1/4}$	.980944
$v^{1/12}$	.993607
$1+i$	1.080000
$(1+i)^{1/2}$	1.039230
$(1+i)^{1/4}$	1.019427
$(1+i)^{1/12}$	1.006434
$i/i^{(2)}$	1.019615
$i/i^{(4)}$	1.029519
$i/i^{(12)}$	1.036157
i/δ	1.039487
$i/d^{(2)}$	1.059615
$i/d^{(4)}$	1.049519
$i/d^{(12)}$	1.042824
i/δ	1.039487

n	v^n	$(1+i)^n$	$a_{\overline{n}\|}$	$s_{\overline{n}\|}$	$1/s_{\overline{n}\|}$
1	.92593	1.08000	.9259	1.0000	1.000000
2	.85734	1.16640	1.7833	2.0800	.480769
3	.79383	1.25971	2.5771	3.2464	.308034
4	.73503	1.36049	3.3121	4.5061	.221921
5	.68058	1.46933	3.9927	5.8666	.170456
6	.63017	1.58687	4.6229	7.3359	.136315
7	.58349	1.71382	5.2064	8.9228	.112072
8	.54027	1.85093	5.7466	10.6366	.094015
9	.50025	1.99900	6.2469	12.4876	.080080
10	.46319	2.15892	6.7101	14.4866	.069029
11	.42888	2.33164	7.1390	16.6455	.060076
12	.39711	2.51817	7.5361	18.9771	.052695
13	.36770	2.71962	7.9038	21.4953	.046522
14	.34046	2.93719	8.2442	24.2149	.041297
15	.31524	3.17217	8.5595	27.1521	.036830
16	.29189	3.42594	8.8514	30.3243	.032977
17	.27027	3.70002	9.1216	33.7502	.029629
18	.25025	3.99602	9.3719	37.4502	.026702
19	.23171	4.31570	9.6036	41.4463	.024128
20	.21455	4.66096	9.8181	45.7620	.021852
21	.19866	5.03383	10.0168	50.4229	.019832
22	.18394	5.43654	10.2007	55.4568	.018032
23	.17032	5.87146	10.3711	60.8933	.016422
24	.15770	6.34118	10.5288	66.7648	.014978
25	.14602	6.84848	10.6748	73.1059	.013679
26	.13520	7.39635	10.8100	79.9544	.012507
27	.12519	7.98806	10.9352	87.3508	.011448
28	.11591	8.62711	11.0511	95.3388	.010489
29	.10733	9.31727	11.1584	103.9659	.009619
30	.09938	10.06266	11.2578	113.2832	.008827
31	.09202	10.86767	11.3498	123.3459	.008107
32	.08520	11.73708	11.4350	134.2135	.007451
33	.07889	12.67605	11.5139	145.9506	.006852
34	.07305	13.69013	11.5869	158.6267	.006304
35	.06763	14.78534	11.6546	172.3168	.005803
36	.06262	15.96817	11.7172	187.1021	.005345
37	.05799	17.24563	11.7752	203.0703	.004924
38	.05369	18.62528	11.8289	220.3159	.004539
39	.04971	20.11530	11.8786	238.9412	.004185
40	.04603	21.72452	11.9246	259.0565	.003860
41	.04262	23.46248	11.9672	280.7810	.003561
42	.03946	25.33948	12.0067	304.2435	.003287
43	.03654	27.36664	12.0432	329.5830	.003034
44	.03383	29.55597	12.0771	356.9496	.002802
45	.03133	31.92045	12.1084	386.5056	.002587
46	.02901	34.47409	12.1374	418.4261	.002390
47	.02686	37.23201	12.1643	452.9002	.002208
48	.02487	40.21057	12.1891	490.1322	.002040
49	.02303	43.42742	12.2122	530.3427	.001886
50	.02132	46.90161	12.2335	573.7702	.001743

INTEREST TABLES AT 12%

Constants	
Function	Value
i	.120000
$i^{(2)}$	.116601
$i^{(4)}$	.114949
$i^{(12)}$	.113866
δ	.113329
d	.107143
$d^{(2)}$	.110178
$d^{(4)}$	.111738
$d^{(12)}$	.112795
δ	.113329
v	.892857
$v^{1/2}$	.944911
$v^{1/4}$	.972065
$v^{1/12}$	.990600
$1+i$	1.120000
$(1+i)^{1/2}$	1.058301
$(1+i)^{1/4}$	1.028737
$(1+i)^{1/12}$	1.009489
$i/i^{(2)}$	1.029150
$i/i^{(4)}$	1.043938
$i/i^{(12)}$	1.053875
i/δ	1.058867
$i/d^{(2)}$	1.089150
$i/d^{(4)}$	1.073938
$i/d^{(12)}$	1.063875
i/δ	1.058867

n	v^n	$(1+i)^n$	$a_{\overline{n}\|}$	$s_{\overline{n}\|}$	$1/s_{\overline{n}\|}$
1	.89286	1.12000	.8929	1.0000	1.000000
2	.79719	1.25440	1.6901	2.1200	.471698
3	.71178	1.40493	2.4018	3.3744	.296349
4	.63552	1.57352	3.0373	4.7793	.209234
5	.56743	1.76234	3.6048	6.3528	.157410
6	.50663	1.97382	4.1114	8.1152	.123226
7	.45235	2.21068	4.5638	10.0890	.099118
8	.40388	2.47596	4.9676	12.2997	.081303
9	.36061	2.77308	5.3282	14.7757	.067679
10	.32197	3.10585	5.6502	17.5487	.056984
11	.28748	3.47855	5.9377	20.6546	.048415
12	.25668	3.89598	6.1944	24.1331	.041437
13	.22917	4.36349	6.4235	28.0291	.035677
14	.20462	4.88711	6.6282	32.3926	.030871
15	.18270	5.47357	6.8109	37.2797	.026824
16	.16312	6.13039	6.9740	42.7533	.023390
17	.14564	6.86604	7.1196	48.8837	.020457
18	.13004	7.68997	7.2497	55.7497	.017937
19	.11611	8.61276	7.3658	63.4397	.015763
20	.10367	9.64629	7.4694	72.0524	.013879
21	.09256	10.80385	7.5620	81.6987	.012240
22	.08264	12.10031	7.6446	92.5026	.010811
23	.07379	13.55235	7.7184	104.6029	.009560
24	.06588	15.17863	7.7843	118.1552	.008463
25	.05882	17.00006	7.8431	133.3339	.007500
26	.05252	19.04007	7.8957	150.3339	.006652
27	.04689	21.32488	7.9426	169.3740	.005904
28	.04187	23.88387	7.9844	190.6989	.005244
29	.03738	26.74993	8.0218	214.5828	.004660
30	.03338	29.95992	8.0552	241.3327	.004144
31	.02980	33.55511	8.0850	271.2926	.003686
32	.02661	37.58173	8.1116	304.8477	.003280
33	.02376	42.09153	8.1354	342.4294	.002920
34	.02121	47.14252	8.1566	384.5210	.002601
35	.01894	52.79962	8.1755	431.6635	.002317
36	.01691	59.13557	8.1924	484.4631	.002064
37	.01510	66.23184	8.2075	543.5987	.001840
38	.01348	74.17966	8.2210	609.8305	.001640
39	.01204	83.08122	8.2330	684.0102	.001462
40	.01075	93.05097	8.2438	767.0914	.001304
41	.00960	104.21709	8.2534	860.1424	.001163
42	.00857	116.72314	8.2619	964.3595	.001037
43	.00765	130.72991	8.2696	1081.0826	.000925
44	.00683	146.41750	8.2764	1211.8125	.000825
45	.00610	163.98760	8.2825	1358.2300	.000736
46	.00544	183.66612	8.2880	1522.2176	.000657
47	.00486	205.70605	8.2928	1705.8838	.000586
48	.00434	230.39078	8.2972	1911.5898	.000523
49	.00388	258.03767	8.3010	2141.9806	.000467
50	.00346	289.00219	8.3045	2400.0182	.000417

Appendix II

Table numbering the days of the year

For leap years the number of the day is one greater than the tabular number after February 28													
Day of Month	January	February	March	April	May	June	July	August	September	October	November	December	Day of Month
1	1	32	60	91	121	152	182	213	244	274	305	335	1
2	2	33	61	92	122	153	183	214	245	275	306	336	2
3	3	34	62	93	123	154	184	215	246	276	307	337	3
4	4	35	63	94	124	155	185	216	247	277	308	338	4
5	5	36	64	95	125	156	186	217	248	278	309	339	5
6	6	37	65	96	126	157	187	218	249	279	310	340	6
7	7	38	66	97	127	158	188	219	250	280	311	341	7
8	8	39	67	98	128	159	189	220	251	281	312	342	8
9	9	40	68	99	129	160	190	221	252	282	313	343	9
10	10	41	69	100	130	161	191	222	253	283	314	344	10
11	11	42	70	101	131	162	192	223	254	284	315	345	11
12	12	43	71	102	132	163	193	224	255	285	316	346	12
13	13	44	72	103	133	164	194	225	256	286	317	347	13
14	14	45	73	104	134	165	195	226	257	287	318	348	14
15	15	46	74	105	135	166	196	227	258	288	319	349	15
16	16	47	75	106	136	167	197	228	259	289	320	350	16
17	17	48	76	107	137	168	198	229	260	290	321	351	17
18	18	49	77	108	138	169	199	230	261	291	322	352	18
19	19	50	78	109	139	170	200	231	262	292	323	353	19
20	20	51	79	110	140	171	201	232	263	293	324	354	20
21	21	52	80	111	141	172	202	233	264	294	325	355	21
22	22	53	81	112	142	173	203	234	265	295	326	356	22
23	23	54	82	113	143	174	204	235	266	296	327	357	23
24	24	55	83	114	144	175	205	236	267	297	328	358	24
25	25	56	84	115	145	176	206	237	268	298	329	359	25
26	26	57	85	116	146	177	207	238	269	299	330	360	26
27	27	58	86	117	147	178	208	239	270	300	331	361	27
28	28	59	87	118	148	179	209	240	271	301	332	362	28
29	29		88	119	149	180	210	241	272	302	333	363	29
30	30		89	120	150	181	211	242	273	303	334	364	30
31	31		90		151		212	243		304		365	31

Appendix IV

Statistical background

A. Moments

1. The mean of a random variable X, denoted by μ_x, is the first moment about the origin
$$\mu_x = E[X].$$
2. The variance of a random variable X, denoted by σ_x^2 or var$[X]$, is the second moment about the mean
$$\sigma_x^2 = \text{var}[X] = E[(X - \mu)^2] = E[X^2] - \{E[X]\}^2.$$
3. The standard deviation of a random variable X, denoted by σ_x, is the square root of the variance.
4. The covariance of two random variables X and Y, denoted by σ_{xy} or cov$[X, Y]$, is defined as
$$\sigma_{xy} = \text{cov}[X, Y] = E\left[(X - \mu_x)(Y - \mu_y)\right] = E[XY] - E[X]E[Y].$$
5. If the random variables X and Y are independent, then $E[XY] = E[X]E[Y]$ and $\sigma_{xy} = \text{cov}[X, Y] = 0$.
6. The mean of $aX + bY$, where a and b are constants, is given by
$$a\mu_x + b\mu_y.$$
7. The variance of $aX + bY$, where a and b are constants, is given by
$$a^2\sigma_x^2 + b^2\sigma_y^2 + 2ab\sigma_{xy}.$$

B. Distributions

1. *Binomial distribution*
A discrete distribution defined by
$$f(x) = \binom{n}{x} p^x(1 - p)^{n-x}$$

where n is a positive integer, $0 < p < 1$, and $x = 0, 1, 2, \ldots, n$. The moments are:

$$\text{mean} = np$$

and

$$\text{variance} = npq.$$

2. *Uniform distribution*
A continuous distribution defined by

$$f(x) = \frac{1}{b - a}$$

where $a \leq x \leq b$. The moments are:

$$\text{mean} = \frac{a + b}{2}$$

and

$$\text{variance} = \frac{(b - a)^2}{12}.$$

3. *Normal distribution*
A continuous distribution defined by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

where $-\infty < \mu < \infty$, $\sigma > 0$, and $-\infty < x < \infty$. The moments are:

$$\text{mean} = \mu$$

and

$$\text{variance} = \sigma^2.$$

Values of the cumulative distribution function are tabulated and used in finding probabilities from this distribution.

4. *Lognormal distribution*
If $Y = \log_e X$ and Y has a normal distribution, then $X > 0$ has a lognormal distribution. The moments are:

$$\text{mean} = e^{\mu + \frac{\sigma^2}{2}}$$

and

$$\text{variance} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1).$$

Note that the mean $\neq \mu$ and the variance $\neq \sigma^2$.

5. This method is very simple to apply on a computer and convergence is guaranteed if $f(x)$ is continuous. However, the rate of convergence is rather slow.

C. Successive inverse interpolation

1. This method also requires two starting values, x_0 and x_1, which have functional values of opposite sign.
2. The iteration formula is given by

$$x_{n+2} = \frac{x_n f(x_{n+1}) - x_{n+1} f(x_n)}{f(x_{n+1}) - f(x_n)}.$$

3. Two variations exist in applying this method after the first iteration. In the first variation the two most recently computed values of x for which the values of $f(x)$ are of opposite sign are used. In the second variation the two most recently computed values of x are used regardless of the sign of $f(x)$.

D. Newton-Raphson method

1. The iteration formula is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

2. For this method $g'(r) = 0$, which produces an extremely fast rate of convergence called "second-order convergence."
3. This method does require that $f'(x)$ can be computed and is non-zero.

Appendix VI

Further analysis of varying annuities

From the subject of numerical analysis, the first difference is defined by

$$\Delta f(t) = f(t+1) - f(t).$$

Higher order differences can be developed by successively applying the above formula.

A formula for summation by parts can be derived which is analogous to integration by parts

$$\sum_{t=1}^{n} v^t f(t) = \left[\frac{v^t}{v-1} \left\{ 1 - \frac{v\Delta}{v-1} + \frac{v^2\Delta^2}{(v-1)^2} - \frac{v^3\Delta^3}{(v-1)^3} + \cdots \right\} f(t) \right]_{t=1}^{t=n+1}$$

$$= \left[\frac{v^{t-1}}{i} \left\{ 1 + \frac{\Delta}{i} + \frac{\Delta^2}{i^2} + \frac{\Delta^3}{i^3} + \cdots \right\} f(t) \right]_{t=n+1}^{t=1}$$

This formula will give practical results whenever higher order differences past a certain point can be safely ignored. In particular, if $f(t)$ is a polynomial of degree m, then $(m+1)$th and higher differences are all zero.

Thus, this formula can be used to find the present value of varying annuities whose payments follow a polynomial.

Month	*Principal*	*Interest*	*Balance*	*Month*	*Principal*	*Interest*	*Balance*
121	119.75	757.82	90,818.60	181	197.03	680.54	81,468.26
122	120.75	756.82	90,697.85	182	198.67	678.90	81,269.59
123	121.75	755.82	90,576.10	183	200.32	677.25	81,069.27
124	122.77	754.80	90,453.33	184	201.99	675.58	80,867.28
125	123.79	753.78	90,329.54	185	203.68	673.89	80,663.60
126	124.82	752.75	90,204.72	186	205.37	672.20	80,458.23
127	125.86	751.71	90,078.86	187	207.08	670.49	80,251.15
128	126.91	750.66	89,951.95	188	208.81	668.76	80,042.34
129	127.97	749.60	89,823.98	189	210.55	667.02	79,831.79
130	129.04	748.53	89,694.94	190	212.31	665.26	79,619.48
131	130.11	747.46	89,564.83	191	214.07	663.50	79,405.41
132	131.20	746.37	89,433.63	192	215.86	661.71	79,189.55
133	132.29	745.28	89,301.34	193	217.66	659.91	78,971.89
134	133.39	744.18	89,167.95	194	219.47	658.10	78,752.42
135	134.50	743.07	89,033.45	195	221.30	656.27	78,531.12
136	135.62	741.95	88,897.83	196	223.14	654.43	78,307.98
137	136.75	740.82	88,761.08	197	225.00	652.57	78,082.98
138	137.89	739.68	88,623.19	198	226.88	650.69	77,856.10
139	139.04	738.53	88,484.15	199	228.77	648.80	77,627.33
140	140.20	737.37	88,343.95	200	230.68	646.89	77,396.65
141	141.37	736.20	88,202.58	201	232.60	644.97	77,164.05
142	142.55	735.02	88,060.03	202	234.54	643.03	76,929.51
143	143.74	733.83	87,916.29	203	236.49	641.08	76,693.02
144	144.93	732.64	87,771.36	204	238.46	639.11	76,454.56
145	146.14	731.43	87,625.22	205	240.45	637.12	76,214.11
146	147.36	730.21	87,477.86	206	242.45	635.12	75,971.66
147	148.59	728.98	87,329.27	207	244.47	633.10	75,727.19
148	149.83	727.74	87,179.44	208	246.51	631.06	75,480.68
149	151.07	726.50	87,028.37	209	248.56	629.01	75,232.12
150	152.33	725.24	86,876.04	210	250.64	626.93	74,981.48
151	153.60	723.97	86,722.44	211	252.72	624.85	74,728.76
152	154.88	722.69	86,567.56	212	254.83	622.74	74,473.93
153	156.17	721.40	86,411.39	213	256.95	620.62	74,216.98
154	157.48	720.09	86,253.91	214	259.10	618.47	73,957.88
155	158.79	718.78	86,095.12	215	261.25	616.32	73,696.63
156	160.11	717.46	85,935.01	216	263.43	614.14	73,433.20
157	161.44	716.13	85,773.57	217	265.63	611.94	73,167.57
158	162.79	714.78	85,610.78	218	267.84	609.73	72,899.73
159	164.15	713.42	85,446.63	219	270.07	607.50	72,629.66
160	165.51	712.06	85,281.12	220	272.32	605.25	72,357.34
161	166.89	710.68	85,114.23	221	274.59	602.98	72,082.75
162	168.28	709.29	84,945.95	222	276.88	600.69	71,805.87
163	169.69	707.88	84,776.26	223	279.19	598.38	71,526.68
164	171.10	706.47	84,605.16	224	281.51	596.06	71,245.17
165	172.53	705.04	84,432.63	225	283.86	593.71	70,961.31
166	173.96	703.61	84,258.67	226	286.23	591.34	70,675.08
167	175.41	702.16	84,083.26	227	288.61	588.96	70,386.47
168	176.88	700.69	83,906.38	228	291.02	586.55	70,095.45
169	178.35	699.22	83,728.03	229	293.44	584.13	69,802.01
170	179.84	697.73	83,548.19	230	295.89	581.68	69,506.12
171	181.34	696.23	83,366.85	231	298.35	579.22	69,207.77
172	182.85	694.72	83,184.00	232	300.84	576.73	68,906.93
173	184.37	693.20	82,999.63	233	303.35	574.22	68,603.58
174	185.91	691.66	82,813.72	234	305.87	571.70	68,297.71
175	187.46	690.11	82,626.26	235	308.42	569.15	67,989.29
176	189.02	688.55	82,437.24	236	310.99	566.58	67,678.30
177	190.59	686.98	82,246.65	237	313.58	563.99	67,364.72
178	192.18	685.39	82,054.47	238	316.20	561.37	67,048.52
179	193.78	683.79	81,860.69	239	318.83	558.74	66,729.69
180	195.40	682.17	81,665.29	240	321.49	556.08	66,408.20

Month	*Principal*	*Interest*	*Balance*	*Month*	*Principal*	*Interest*	*Balance*
241	324.17	553.40	66,084.03	301	533.36	344.21	40,772.20
242	326.87	550.70	65,757.16	302	537.80	339.77	40,234.40
243	329.59	547.98	65,427.57	303	542.28	335.29	39,692.12
244	332.34	545.23	65,095.23	304	546.80	330.77	39,145.32
245	335.11	542.46	64,760.12	305	551.36	326.21	38,593.96
246	337.90	539.67	64,422.22	306	555.95	321.62	38,038.01
247	340.72	536.85	64,081.50	307	560.59	316.98	37,477.42
248	343.56	534.01	63,737.94	308	565.26	312.31	36,912.16
249	346.42	531.15	63,391.52	309	569.97	307.60	36,342.19
250	349.31	528.26	63,042.21	310	574.72	302.85	35,767.47
251	352.22	525.35	62,689.99	311	579.51	298.06	35,187.96
252	355.15	522.42	62,334.84	312	584.34	293.23	34,603.62
253	358.11	519.46	61,976.73	313	589.21	288.36	34,014.41
254	361.10	516.47	61,615.63	314	594.12	283.45	33,420.29
255	364.11	513.46	61,251.52	315	599.07	278.50	32,821.22
256	367.14	510.43	60,884.38	316	604.06	273.51	32,217.16
257	370.20	507.37	60,514.18	317	609.09	268.48	31,608.07
258	373.29	504.28	60,140.89	318	614.17	263.40	30,993.90
259	376.40	501.17	59,764.49	319	619.29	258.28	30,374.61
260	379.53	498.04	59,384.96	320	624.45	253.12	29,750.16
261	382.70	494.87	59,002.26	321	629.65	247.92	29,120.51
262	385.88	491.69	58,616.38	322	634.90	242.67	28,485.61
263	389.10	488.47	58,227.28	323	640.19	237.38	27,845.42
264	392.34	485.23	57,834.94	324	645.52	232.05	27,199.90
265	395.61	481.96	57,439.33	325	650.90	226.67	26,549.00
266	398.91	478.66	57,040.42	326	656.33	221.24	25,892.67
267	402.23	475.34	56,638.19	327	661.80	215.77	25,230.87
268	405.59	471.98	56,232.60	328	667.31	210.26	24,563.56
269	408.96	468.61	55,823.64	329	672.87	204.70	23,890.69
270	412.37	465.20	55,411.27	330	678.48	199.09	23,212.21
271	415.81	461.76	54,995.46	331	684.13	193.44	22,528.08
272	419.27	458.30	54,576.19	332	689.84	187.73	21,838.24
273	422.77	454.80	54,153.42	333	695.58	181.99	21,142.66
274	426.29	451.28	53,727.13	334	701.38	176.19	20,441.28
275	429.84	447.73	53,297.29	335	707.23	170.34	19,734.05
276	433.43	444.14	52,863.86	336	713.12	164.45	19,020.93
277	437.04	440.53	52,426.82	337	719.06	158.51	18,301.87
278	440.68	436.89	51,986.14	338	725.05	152.52	17,576.82
279	444.35	433.22	51,541.79	339	731.10	146.47	16,845.72
280	448.06	429.51	51,093.73	340	737.19	140.38	16,108.53
281	451.79	425.78	50,641.94	341	743.33	134.24	15,365.20
282	455.55	422.02	50,186.39	342	749.53	128.04	14,615.67
283	459.35	418.22	49,727.04	343	755.77	121.80	13,859.90
284	463.18	414.39	49,263.86	344	762.07	115.50	13,097.83
285	467.04	410.53	48,796.82	345	768.42	109.15	12,329.41
286	470.93	406.64	48,325.89	346	774.82	102.75	11,554.59
287	474.85	402.72	47,851.04	347	781.28	96.29	10,773.31
288	478.81	398.76	47,372.23	348	787.79	89.78	9,985.52
289	482.80	394.77	46,889.43	349	794.36	83.21	9,191.16
290	486.82	390.75	46,402.61	350	800.98	76.59	8,390.18
291	490.88	386.69	45,911.73	351	807.65	69.92	7,582.53
292	494.97	382.60	45,416.76	352	814.38	63.19	6,768.15
293	499.10	378.47	44,917.66	353	821.17	56.40	5,946.98
294	503.26	374.31	44,414.40	354	828.01	49.56	5,118.97
295	507.45	370.12	43,906.95	355	834.91	42.66	4,284.06
296	511.68	365.89	43,395.27	356	841.87	35.70	3,442.19
297	515.94	361.63	42,879.33	357	848.89	28.68	2,593.30
298	520.24	357.33	42,359.09	358	855.96	21.61	1,737.34
299	524.58	352.99	41,834.51	359	863.09	14.48	874.25
300	528.95	348.62	41,305.56	360	870.28	7.29	3.97

Appendix IX

Derivation of the variance of an annuity

It is desired to find the variance of the sum

$$s = x_1 + x_1x_2 + \cdot \cdot \cdot + x_1x_2 \ldots x_n$$

where $x_1, x_2, \ldots, x_n$ are independent and identically distributed random variables Both formula (3.34) for $a_{\overline{n}|}$ and formula (3.36) for $\ddot{s}_{\overline{n}|}$ are of this form.

Let m_1 and m_2 be the first and second moments about the origin, i.e.

$$E[x_k] = m_1 \quad \text{and} \quad E\left[x_k^2\right] = m_2$$

for $k = 1, 2, \ldots, n$. From independence we have

$$E[s] = \sum_{k=1}^{n} m_1^k$$

which we will denote by $s_n(m_1)$.

The formula which we wish to prove is

$$\text{var}[s] = \frac{m_2 + m_1}{m_2 - m_1} s_n(m_2) - \frac{2m_2}{m_2 - m_1} s_n(m_1) - \left[s_n(m_1)\right]^2.$$

The proof is by mathematical induction.

Let $n = 1$. We know that

$$\text{var}[s] = \text{var}[x_1] = m_2 - m_1^2.$$

The right-hand side of the formula to be proven is

$$\frac{m_2 + m_1}{m_2 - m_1} m_2 - \frac{2m_2}{m_2 - m_1} m_1 - m_1^2 = m_2 - m_1^2.$$

Thus, the formula holds for $n = 1$.

Now assume that the formula holds for $n - 1$. Define

$$t = x_1 + x_1x_2 + \cdot \cdot \cdot + x_1x_2 \ldots x_{n-1}.$$

We have

$$\begin{aligned} \text{var}[s] &= \text{var}\left[t + x_1x_2 \ldots x_n\right] \\ &= \text{var}[t] + \text{var}\left[x_1x_2 \ldots x_n\right] + 2\text{cov}\left[t, x_1x_2 \ldots x_n\right]. \end{aligned}$$

We need to evaluate these three terms.

From the induction assumption

$$\text{var}[t] = \frac{m_2 + m_1}{m_2 - m_1} s_{n-1}(m_2) - \frac{2m_2}{m_2 - m_1} s_{n-1}(m_1) - \left[s_{n-1}(m_1)\right]^2$$

$$= \frac{m_2 + m_1}{m_2 - m_1}\left(m_2 + m_2^2 + \cdots + m_2^{n-1}\right) \quad \text{(A)}$$

$$- \frac{2m_2}{m_2 - m_1}\left(m_1 + m_1^2 + \cdots + m_1^{n-1}\right) \quad \text{(B)}$$

$$- \left(m_1 + m_1^2 + \cdots + m_1^{n-1}\right)^2 \quad \text{(C)}$$

From independence

$$\text{var}\left[x_1 x_2 \ldots x_n\right] = m_2^n - m_1^{2n}$$

$$= \frac{m_2^{n+1} - m_2^n m_1}{m_2 - m_1} - m_1^{2n}.$$

(D) (E)

For the covariance term we have

$$2\,\text{cov}\left[t,\ x_1 x_2 \ldots x_n\right] = 2E\left[t x_1 x_2 \ldots x_n\right] - 2E[t]E\left[x_1 x_2 \ldots x_n\right]$$

$$= 2\left(m_2 m_1^{n-1} + m_2^2 m_1^{n-2} + \cdots + m_2^{n-1} m_1\right) - 2\left(m_1 + m_1^2 + \cdots + m_1^{n-1}\right)m$$

$$= 2 \cdot \frac{m_2^n m_1 - m_1^n m_2}{m_2 - m_1} - 2m_1^n\left(m_1 + m_1^2 + \cdots + m_1^{n-1}\right)$$

$$= \frac{2m_1 m_2^n}{m_2 - m_1} - \frac{2m_1^n m_2}{m_2 - m_1} - 2m_1^n\left(m_1 + m_1^2 + \cdots + m_1^{n-1}\right)$$

(F) (G) (H)

We need to show that the sum of the three terms gives us our induction hypothesis. The reader should verify that (A) + (D) + (F) gives the first term in the formula to be proven; (B) + (G) gives the second term; and (C) + (E) + (H) gives the third term.

Thus, the proof by mathematical induction is complete.

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33. 8

34. 20

35. r^{-4}

38. $d < d^{(m)} < \delta < i^{(m)} < i$

40. *a)* $\log_e a + 2t\log_e b + c^t \log_e c \log_e d$ *b)* Formula (1.25)

42. 5

43. 14/205

44. $\dfrac{\log_e i - \log_e \delta}{\delta}$

45. $e^{.01} - 1$

46. $e^{.038}$

47. $.804264^{-1/3} - 1$

48. *a)* $(1 + r)^{n(n+1)/2}(1 + i)^n$ *b)* $(1 + r)^{(n+1)/2}(1 + i) - 1$

49. .82

50. $e^{.3}$

51. $2/n$

52. 9.48%

53. *b)* $\dfrac{\delta^3}{3!} + \dfrac{\delta^5}{5!} + \dfrac{\delta^7}{7!} + \cdots$

55. *a)* $(1 + i)^{-2}$ *b)* $(1 + i)^{-1}$ *c)* $(1 - d)^{(1-m)/m}$

d) $-v^{-1}$ *e)* $e^{-\delta}$

56. *a)* $d + d^2 + d^3 + d^4 + \cdots$

b) $i - i^2 + i^3 - i^4 + \cdots$

c) $1 - \delta + \dfrac{\delta^2}{2!} - \dfrac{\delta^3}{3!} + \cdots$

d) $i - \dfrac{m-1}{2!m} i^2 + \dfrac{(m-1)(2m-1)}{3!m^2} i^3 - \cdots$

e) $d + \dfrac{d^2}{2} + \dfrac{d^3}{3} + \dfrac{d^4}{4} + \cdots$

59. *a)* (1) $e^{at+bt^2/2}$ (2) $e^{(a-b/2)+bn}$

b) (1) $e^{a(b^t-1)/\log_e b}$ (2) $e^{a(b-1)b^{n-1}/\log_e b}$

Chapter 2

1. \$10,402.36
2. *a*) \$4225.27 *b*) \$4225.46
3. $\dfrac{\log_e \delta - \log_e d}{\delta}$
4. 120/121
5. *a*) 1340 *b*) 1321
6. *a*) \$101.92 *b*) \$100.00 *c*) \$103.33
7. *c*) Invest for the month of February
8. \$1593.00
9. \$917.76
10. \$483.11
11. *a*) \$275 *b*) \$260
12. $\dfrac{1}{\delta}\left[\log_e(s_1 + s_2 + \cdots + s_n) - \log_e(s_1 v^{t_1} + s_2 v^{t_2} + \cdots + s_n v^{t_n})\right]$
13. 36.4 periods
14. 6.25 years
15. $\dfrac{2n^2 + n}{3}$
16. 114
17. .30
18. $144 \log_e 1.01$
19. 7.46%
20. $\dfrac{\sqrt{19} - 4}{3}$
21. 1.5%
22. $\dfrac{\log_e 2}{50}$
23. 4%
25. *a*) 16% *b*) 17.74%
26. *a*) 1.0124 *b*) .9938
27. *a*) 7.00% *b*) 7.71% *c*) 7.53%
28. \$1540.34

8. $300\dfrac{1-(.9925)^{120}}{1-(.9925)^{6}}$

9. 20%

10. $100\dfrac{1-e^{-.4}}{1-e^{-.02}}$

11. 1.95

12. $\dfrac{1-v^{48}}{1-v^{4/3}}$

13. \$11,466

18. $\dfrac{i^{(2)}}{d^{(12)}}a_{\overline{n}|}^{(2)}$

20. $\dfrac{\log_e 20[1-(1-d)^{.5}]}{\log_e(1-d)}$

21. 1/30

22. $\dfrac{1-v^{36}}{1-v^{3/4}}$

23. $1-\dfrac{1}{\delta}\log_e\dfrac{i}{\delta}$

25. $\log_e(n+1)$

26. 27.47 years

27. 1/6

30. $6a_{\overline{20}|}+\dfrac{a_{\overline{20}|}-20v^{20}}{i}$

33. $a_{\overline{n}|}/d$

34. 66

35. 1/21

36. $6250-325A$

37. \$16,607

38. \$7851

40. *a*) 3 *b*) 25/12

43. 48

45. 112.59

46. 3

47. 156.25

48. $\dfrac{1}{\delta_i - \delta_k}$

49. *a)* $\int_0^n (n-t)v^t dt$ *b)* $\dfrac{n - \bar{a}_{\overline{n}|}}{\delta}$

50. 16,020

51. 84.5

52. *a)* (2) $-\dfrac{n(n+1)}{2}$ *b)* (2) $-\dfrac{n^2}{2}$

54. 40

55. *a)* $\dfrac{n(1+i) - 2\ddot{a}_{\overline{n}|} + nv^n}{i^2}$ *b)* $\dfrac{\frac{1}{2}in(n+1) - n + a_{\overline{n}|}}{i^2}$

58. *a)* $\dfrac{q}{p-q}$ *b)* $\dfrac{2q}{p-q}$

60. $\dfrac{a^3}{(2a-b)^2}\left[2a - b - (b-a)\log_e \dfrac{a}{b-a}\right]$

Chapter 5

1. *a)* $1000(a_{\overline{9}|} + 4a_{\overline{1}|} + 10)$ *b)* $1000\left[2(a_{\overline{10}|} - a_{\overline{5}|}) + (Ia)_{\overline{10}|} - (Ia)_{\overline{5}|}\right]$

2. 3

3. \$20,206

4. *a)* 75.05 *b)* -57.85

5. $i_{s+1} = i_s + \dfrac{\sum_{t=0}^{n} v_s^t R_t}{\sum_{t=1}^{n} t v_s^{t+1} R_t}$

6. *a)* 3 *b)* Yes

7. .02

9. Any X, $0 < X < 100$

11. 6.16%

12. \$4450

14. \$8439

32. \$1344.89

33. \$966.08

34. \$156

37. \$78.20

38. \$908.87

40. *a*) \$1287.76 *b*) \$276.24

41. 13

42. 4%

43. 2.8659

45. *a*) $K\bar{a}_{\overline{n-k}|} + (\bar{I}\bar{a})_{\overline{n-k}|}$ *b*) $(\bar{I}\bar{a})_{\overline{n}|}(1+i)^k - (\bar{I}\bar{s})_{\overline{k}|}$

46. *a*) .5 *b*) .375

47. (6:14): $L = B_0 = \int_0^n e^{-\int_0^s \delta_r dr} R_s ds$

(6:15): $B_t^p = \int_t^n e^{-\int_t^s \delta_r dr} R_s ds$

(6:16): $B_t^r = B_0 e^{\int_0^t \delta_r dr} - \int_0^t R_s e^{\int_s^t \delta_r dr} ds$

48. *a*) $\alpha\beta e^{-\beta t}$ *b*) α *c*) $\frac{\alpha\beta}{\beta+\delta}$ *d*) $\frac{\alpha\beta}{\beta+\delta} e^{-\beta t}$

49. \$310

50. *a*) \$79.59 *b*) (1) 15

(2) 1 2/3%

51. \$272.42

54. \$5736

55. *a*) \$757.19 *b*) \$826.40

56. $R\dfrac{a_{\overline{30}|.10} - a_{\overline{15}|.09}}{a_{\overline{20}|.09} - a_{\overline{15}|.09}}$

57. *a*) $\dfrac{(80{,}000/a_{\overline{20}|\ .08})a_{\overline{11}|\ .08} - 5000}{a_{\overline{9}|\ .09}}$

b) $\dfrac{80{,}000(1.09)^9 - (80{,}000/a_{\overline{20}|\ .08})s_{\overline{9}|\ .09} - 5000}{a_{\overline{9}|\ .09}}$

58. \$571

Chapter 7

1. *a*) \$385.54 *b*) \$422.41 *c*) 9.56%
2. \$844.77
3. *a*) 7.91% b) 8.51%
4. 1.0332
5. \$115.87
6. *a*) 8.40% *b*) 8.00% *c*) 9.14% *d*) 10.00%
7. *a*) −7.72% *b*) −12.46%
8. \$794.83
9. \$945
10. \$1200
11. \$1291
12. \$1100
16. $1 - .5p$
17. \$20
18. \$33.98
19. \$573.60
20. *a*)

	b)
1037.17	964.54
1027.88	973.41
1018.59	982.27
1009.29	991.14
1000.00	1000.00

24.

	$B_{1/3}^{f}$	$Fr_{1/3}$	$B_{1/3}^{m}$
Theoretical	980.35	13.12	967.23
Practical	980.62	13.33	967.29
Semi-Theoretical	980.35	13.33	967.02

25. *a*) Theoretical = Semi-Theoretical < Practical

b) Semi-Theoretical < Theoretical

Semi-Theoretical < Practical

Practical $\frac{\leq}{>}$ Theoretical

40. $4.61

41. $365.63

42. $116,500

43. 31.33%

44. $\dfrac{7+4m}{50m}$

Chapter 9

1. 1.00

2. .82

3. 12.9%

4. 116.8%

5. *a*) $25,633 *b*) $23,736 *c*) $16,658

6. *a*) Option A *b*) Option B

7. 8.07%

8. $767.55

9. *a*) 720 *b*) 240

10. 13.89%

11. .00685

12. *b*) c *c*) $1-e^{-c}$ *d*) $1-e^{-cn}$

13. *a*) 1,161,400 *b*) 16,600 *c*) 7.77%

14. *a*) $141,500 *b*) $128,300

15. 7.91%

16. *a*) $946.93 *b*) $1036.53 *c*) 7.975% *d*) 7.953% *f*) Yes

17. *a*) $857.33 *b*) $1000.00

18. 6.4646

19. *a*) 9.64% *b*) 10.51%

20. 26.45%

21. Minimum = $128,614

    Maximum = $129,470

23. $-\dfrac{1}{\delta}\log_e\left[\dfrac{(v+2v^2)a_{\overline{20}|}}{30a_{\overline{2}|}}\right]$

24. 27

25. $\dfrac{v(\bar{I}\,\bar{a})_{\overline{n}|}}{\bar{a}_{\overline{n}|}}$

28. *a*) \$3616 *b*) 2.2035 *c*) 1.7628 *d*) .6046

30. \$1591

32. $a(i) = v^2$ $b(i) = \bar{v}(\bar{v} + v)$

33. *a*) (1) -1.8012 (2) 0 (3) 1.8343

b) (1) 1.8854 (2) 0 (3) -1.7531

34. *a*) 0 *b*) 94.307 *c*) 312.5

35. $\frac{1}{3}(n + 1)(n + 2)$

36. 1250

38. \$13,223 in 1-year zero coupon bonds
\$15,061 in 3-year zero coupon bonds
\$9624 in 5-year zero coupons bonds

39. \$9091 in 1-year zero coupon bonds
\$8265 in 2-year zero coupon bonds
\$7513 in 3-year zero coupon bonds
\$6830 in 4-year zero coupon bonds
\$6209 in 5-year zero coupon bonds

40. *a*) \$179.37 *b*) 12.01%

41. *a*) $0 < p_1 < .6980$ *b*) No solution exists

42. *a*) $.2186 < p_1 < .5931$ *b*) No solution exists

Chapter 10

3. *a*) 8% in all years *b*) 0 in year 1; .01 in year 2; $.01\sqrt{2}$ in year 3

c) \$1294.92 *d*) \$1224.94 *e*) \$1259.71 *f*) \$1259.82 *g*) 26.08

4. *a*) .07997 *b*) .79390 *c*) .16630 *d*) .00735

5. mean = 2.5772
s.d. = .0169

6. *a*) mean = 1.823 s.d. = .058 *b*) mean = 14.121 s.d. = .297 *c*) mean = .549 s.d. = .017 *d*) mean = 7.298 s.d. = .134

Symbol	Section	Description
B_t^r	6.2	retrospective loan balance
B_{t+k}^f	7.5	flat price of a bond
B_{t+k}^m	7.5	market price of a bond
B^c	10.5	value of a callable bond
B^{nc}	10.5	value of a noncallable bond
$\bar{c}$	9.9	convexity
C	7.3	redemption value of a bond
	10.5	value of a call
C_d	10.6	value of call if stock goes down
C_t	5.2	contribution, deposit
C'_k	5.6	contribution, deposit at time t_k
C_m	7.8	redemption value of a serial bond
C_u	10.6	value of call if stock goes up
C'	7.8	summation of redemption values of serial bonds
cov[X, Y]	10.3	covariance of random variables X and Y
d	1.7	effective rate of discount
	1.7	rate of simple discount
	1.7	rate of compound discount
	8.5	factor in the declining balance method of depreciation
	App. III	common difference in an arithmetic progression
d'	8.5	factor in a variation of the declining balance method of depreciation
$\bar{d}$	9.8	duration, Macaulay duration
d_n	1.7	effective rate of discount from time $n-1$ to n
$d^{(m)}$	1.8	nominal rate of discount convertible $m$thly
d_1, d_2	10.5	constants in the Black-Scholes formula
D	6.4	sinking fund deposit
	7.10	dividend on stock
D_t	8.5	depreciation charge
D_1, D_2	2.3	day of date
$(Da)_{\overline{n}\rceil}$	4.6	present value of a decreasing annuity

Symbol	Section	Description
$(Ds)_{\overline{n}\rceil}$	4.6	accumulated value of a decreasing annuity
e	10.4	a measure of equity required to support liabilities
$e(t)$	10.3	error term in AR(1) and AR(2) processes
E	10.5	exercise (striking) price of an option
$E[X]$	10.2	expectation of random variable X
EPV	9.5	expected present value
f	5.9	project financing rate
	9.6	forward rate
$f(x)$	App. IV	probability density function
	3.8	function being solved using iteration
F	7.3	face amount, par value of a bond
F_n	4.6	present value of single payment at time n
Fr	7.3	bond coupon
Fr_k	7.5	accrued bond coupon
$F(x)$	App. IV	cumulative distribution function
g	7.3	modified coupon rate on a bond
$g(x)$	3.8	iteration formula
g_1, g_2	10.3	roots of the characteristic equation for the AR(2) proce
G	7.3	base amount of a bond
G_n	4.6	present value of level perpetuity starting at time n
h	10.6	length of time interval in binomial lattice
H	8.6	periodic charge of an asset
H_n	4.6	present value of increasing perpetuity starting at time n
i	1.3	effective rate of interest
	1.4	rate of simple interest
	1.5	rate of compound interest
	5.2	yield rate, internal rate of return
	5.4	investment rate when j is the reinvestment rate
	6.4	interest rate on loan in sinking fund method
	8.2	annual percentage rate (APR)
	9.4	nominal rate of interest

Symbol	Section	Description
L_t	9.9	cash outflow generated by liabilities
m_1	App. VIII	first moment about the origin
m_2	App. VIII	second moment about the origin
m_1^s	10.2	first moment of $1 + i_t$ about the origin
m_2^s	10.2	second moment of $1 + i_t$ about the origin
m_1^a	10.2	first moment of $(1 + i_t)^{-1}$ about the origin
m_2^a	10.2	second moment of $(1 + i_t)^{-1}$ about the origin
M	8.6	periodic maintenance expense
M_1, M_2	2.3	month of date
n	various	the term of a transaction
N	10.5	cumulative distribution function for the standard normal distribution
p	7.4	premium or discount on a \$1 bond
	9.5	probability bond does not default
	10.6	probability of price increases in binomial lattice
p_t	9.5	probability bond does not default at time t
p_1, p_2	9.10	proportion of funds in different investments in asset/liability matching
P	7.3	price of a bond
	10.5	price of a put
P_m	7.8	price of a serial bond
P'	7.8	summation of prices of serial bonds
P_t	6.3	amount of principal repaid on loan
	7.4	amount of principal adjustment on bond
$\bar{P}_t$	6.7	instantaneous rate at which principal is being repaid
$P(i)$	5.2	net present value
$P(i_*)$	9.6	net present value based on spot rates
q	9.5	probability bond defaults
Q	8.3	expenses at settlement on real estate mortgage that must be reflected in APR

Symbol	Section	Description
r	5.9	project return rate
	7.3	coupon rate on a bond
	9.4	rate of inflation
	10.4	yield rate in CAPM
	App. III	common ratio in a geometric progression
r_e	10.4	return on equity
r_f	10.4	risk-free rate of interest
r_k	10.4	yield rate on a specific security
r_l	10.4	appropriate rate to discount liabilities
r_p	10.4	yield rate on the market portfolio
R	4.3	equivalent payment per interest conversion period
	6.3	level installment payment to repay a loan
	8.5	level periodic return
R_t	5.2	return, withdrawal
	6.6	varying installment payments to repay a loan
	9.9	net receipt in immunization
s	App. III	sum of a progression
	App. VIII	sum of an annuity
s^2	10.2	variance of rate of interest
s_1, s_2	9.10	withdrawal amounts at different times in asset/l matching
		accumulated value of an n-period
$s_{\overline{n}\|}$	3.2	annuity-immediate
$\ddot{s}_{\overline{n}\|}$	3.3	annuity-due
$s^{(m)}_{\overline{n}\|}$	4.4	annuity-immediate payable $m$thly
$\ddot{s}^{(m)}_{\overline{n}\|}$	4.4	annuity-due payable $m$thly
$\bar{s}_{\overline{n}\|}$	4.5	continuous annuity
$s_n(m_1)$	App. VIII	expected value of the sum of an annuity
S	8.5	salvage value of asset after depreciation
	10.5	current stock price
S_r	8.4	sum of the first r positive integers
S_t	App. IX	random variable for the value of an asset

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